# Miscellaneous FOL Examples

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## Contents

# 1 Natural numbers

**theory** *Natural-Numbers*
**imports** *FOL*
**begin**

Theory of the natural numbers: Peano's axioms, primitive recursion. (Modernized version of Larry Paulson's theory "Nat".)

**typedecl** *nat*
**instance** *nat* :: ‹*term*› ⟨*proof*⟩

**axiomatization**
*Zero* :: ‹*nat*› (‹*0*›) **and**
*Suc* :: ‹*nat* => *nat*› **and**
*rec* :: ‹[*nat*, '*a*, [*nat*, '*a*] => '*a*] => '*a*›
**where**
*induct* [*case-names 0 Suc*, *induct type*: *nat*]:
‹*P*(*0*) ==> (!!*x*. *P*(*x*) ==> *P*(*Suc*(*x*))) ==> *P*(*n*)› **and**
*Suc-inject*: ‹*Suc*(*m*) = *Suc*(*n*) ==> *m* = *n*› **and**
*Suc-neq-0*: ‹*Suc*(*m*) = *0* ==> *R*› **and**
*rec-0*: ‹*rec*(*0*, *a*, *f*) = *a*› **and**

*rec-Suc*: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,\ a,\ f))$›

**lemma** *Suc-n-not-n*: ‹$Suc(k) \neq k$›
⟨*proof*⟩

**definition** *add* :: ‹*nat* => *nat* => *nat*›    (**infixl** ‹+› *60*)
  **where** ‹$m + n = rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

**lemma** *add-0* [*simp*]: ‹$0 + n = n$›
  ⟨*proof*⟩

**lemma** *add-Suc* [*simp*]: ‹$Suc(m) + n = Suc(m + n)$›
  ⟨*proof*⟩

**lemma** *add-assoc*: ‹$(k + m) + n = k + (m + n)$›
  ⟨*proof*⟩

**lemma** *add-0-right*: ‹$m + 0 = m$›
  ⟨*proof*⟩

**lemma** *add-Suc-right*: ‹$m + Suc(n) = Suc(m + n)$›
  ⟨*proof*⟩

**lemma**
  **assumes** ‹!!$n.\ f(Suc(n)) = Suc(f(n))$›
  **shows** ‹$f(i + j) = i + f(j)$›
  ⟨*proof*⟩

**end**

# 2 Examples for the manual "Introduction to Isabelle"

**theory** *Intro*
**imports** *FOL*
**begin**

### 2.0.1 Some simple backward proofs

**lemma** *mythm*: ‹$P \vee P \longrightarrow P$›
⟨*proof*⟩

**lemma** ‹$(P \wedge Q) \vee R \longrightarrow (P \vee R)$›
⟨*proof*⟩

Correct version, delaying use of *spec* until last.

**lemma** ‹$(\forall x\ y.\ P(x,y)) \longrightarrow (\forall z\ w.\ P(w,z))$›

$\langle proof \rangle$

### 2.0.2 Demonstration of *fast*

**lemma** ‹$(\exists\, y.\ \forall\, x.\ J(y,x) \longleftrightarrow \neg\ J(x,x)) \longrightarrow \neg\ (\forall\, x.\ \exists\, y.\ \forall\, z.\ J(z,y) \longleftrightarrow \neg\ J(z,x))$›
$\langle proof \rangle$

**lemma** ‹$\forall\, x.\ P(x,f(x)) \longleftrightarrow (\exists\, y.\ (\forall\, z.\ P(z,y) \longrightarrow P(z,f(x))) \wedge P(x,y))$›
$\langle proof \rangle$

### 2.0.3 Derivation of conjunction elimination rule

**lemma**
  **assumes** *major*: ‹$P \wedge Q$›
    **and** *minor*: ‹$[\![P;\ Q]\!] \Longrightarrow R$›
  **shows** ‹$R$›
$\langle proof \rangle$

## 2.1 Derived rules involving definitions

Derivation of negation introduction

**lemma**
  **assumes** ‹$P \Longrightarrow False$›
  **shows** ‹$\neg\ P$›
$\langle proof \rangle$

**lemma**
  **assumes** *major*: ‹$\neg\ P$›
    **and** *minor*: ‹$P$›
  **shows** ‹$R$›
$\langle proof \rangle$

Alternative proof of the result above

**lemma**
  **assumes** *major*: ‹$\neg\ P$›
    **and** *minor*: ‹$P$›
  **shows** ‹$R$›
$\langle proof \rangle$

**end**

# 3 Theory of the natural numbers: Peano's axioms, primitive recursion

**theory** *Nat*
  **imports** *FOL*
**begin**

**typedecl** *nat*
**instance** *nat* :: ‹*term*› ⟨*proof*⟩

**axiomatization**
  *Zero* :: ‹*nat*›  (‹*0*›) **and**
  *Suc* :: ‹*nat* ⇒ *nat*› **and**
  *rec* :: ‹[*nat*, *'a*, [*nat*, *'a*] ⇒ *'a*] ⇒ *'a*›
**where**
  *induct*: ‹$[\![P(0);\ \bigwedge x.\ P(x) \Longrightarrow P(Suc(x))]\!] \Longrightarrow P(n)$› **and**
  *Suc-inject*: ‹$Suc(m)=Suc(n) \Longrightarrow m=n$› **and**
  *Suc-neq-0*: ‹$Suc(m)=0 \Longrightarrow R$› **and**
  *rec-0*: ‹$rec(0,a,f) = a$› **and**
  *rec-Suc*: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,a,f))$›

**definition** *add* :: ‹[*nat*, *nat*] ⇒ *nat*›  (**infixl** ‹+› *60*)
  **where** ‹$m + n \equiv rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

## 3.1 Proofs about the natural numbers

**lemma** *Suc-n-not-n*: ‹$Suc(k) \neq k$›
⟨*proof*⟩

**lemma** ‹$(k+m)+n = k+(m+n)$›
⟨*proof*⟩

**lemma** *add-0* [*simp*]: ‹$0+n = n$›
⟨*proof*⟩

**lemma** *add-Suc* [*simp*]: ‹$Suc(m)+n = Suc(m+n)$›
⟨*proof*⟩

**lemma** *add-assoc*: ‹$(k+m)+n = k+(m+n)$›
⟨*proof*⟩

**lemma** *add-0-right*: ‹$m+0 = m$›
⟨*proof*⟩

**lemma** *add-Suc-right*: ‹$m+Suc(n) = Suc(m+n)$›
⟨*proof*⟩

**lemma**
  **assumes** *prem*: ‹$\bigwedge n.\ f(Suc(n)) = Suc(f(n))$›
  **shows** ‹$f(i+j) = i+f(j)$›
⟨*proof*⟩

**end**

# 4 Theory of the natural numbers: Peano's axioms, primitive recursion

**theory** *Nat-Class*
 **imports** *FOL*
**begin**

This is an abstract version of `Nat.thy`. Instead of axiomatizing a single type *nat*, it defines the class of all these types (up to isomorphism).

Note: The *rec* operator has been made *monomorphic*, because class axioms cannot contain more than one type variable.

**class** *nat* =
 **fixes** *Zero* :: ‹$'a$› (‹*0*›)
  **and** *Suc* :: ‹$'a \Rightarrow 'a$›
  **and** *rec* :: ‹$'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a$›
 **assumes** *induct*: ‹$P(0) \Longrightarrow (\bigwedge x.\ P(x) \Longrightarrow P(Suc(x))) \Longrightarrow P(n)$›
  **and** *Suc-inject*: ‹$Suc(m) = Suc(n) \Longrightarrow m = n$›
  **and** *Suc-neq-Zero*: ‹$Suc(m) = 0 \Longrightarrow R$›
  **and** *rec-Zero*: ‹$rec(0,\ a,\ f) = a$›
  **and** *rec-Suc*: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,\ a,\ f))$›
**begin**

**definition** *add* :: ‹$'a \Rightarrow 'a \Rightarrow 'a$› (**infixl** ‹+› *60*)
 **where** ‹$m + n = rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

**lemma** *Suc-n-not-n*: ‹$Suc(k) \neq (k{::}'a)$›
 ⟨*proof*⟩

**lemma** ‹$(k + m) + n = k + (m + n)$›
 ⟨*proof*⟩

**lemma** *add-Zero* [*simp*]: ‹$0 + n = n$›
 ⟨*proof*⟩

**lemma** *add-Suc* [*simp*]: ‹$Suc(m) + n = Suc(m + n)$›
 ⟨*proof*⟩

**lemma** *add-assoc*: ‹$(k + m) + n = k + (m + n)$›
 ⟨*proof*⟩

**lemma** *add-Zero-right*: ‹$m + 0 = m$›
 ⟨*proof*⟩

**lemma** *add-Suc-right*: ‹$m + Suc(n) = Suc(m + n)$›
 ⟨*proof*⟩

**lemma**
 **assumes** *prem*: ‹$\bigwedge n.\ f(Suc(n)) = Suc(f(n))$›

**shows** ‹$f(i + j) = i + f(j)$›
⟨*proof*⟩

**end**

**end**

# 5 Intuitionistic FOL: Examples from The Foundation of a Generic Theorem Prover

**theory** *Foundation*
**imports** *IFOL*
**begin**

**lemma** ‹$A \wedge B \longrightarrow (C \longrightarrow A \wedge C)$›
⟨*proof*⟩

A form of conj-elimination

**lemma**
**assumes** ‹$A \wedge B$›
**and** ‹$A \Longrightarrow B \Longrightarrow C$›
**shows** ‹$C$›
⟨*proof*⟩

**lemma**
**assumes** ‹$\bigwedge A.\ \neg\ \neg\ A \Longrightarrow A$›
**shows** ‹$B \vee \neg\ B$›
⟨*proof*⟩

**lemma**
**assumes** ‹$\bigwedge A.\ \neg\ \neg\ A \Longrightarrow A$›
**shows** ‹$B \vee \neg\ B$›
⟨*proof*⟩

**lemma**
**assumes** ‹$A \vee \neg\ A$›
**and** ‹$\neg\ \neg\ A$›
**shows** ‹$A$›
⟨*proof*⟩

## 5.1 Examples with quantifiers

**lemma**
**assumes** ‹$\forall z.\ G(z)$›
**shows** ‹$\forall z.\ G(z) \vee H(z)$›
⟨*proof*⟩

**lemma** ‹$\forall x.\ \exists y.\ x = y$›
⟨*proof*⟩

**lemma** ‹$\exists y.\ \forall x.\ x = y$›
⟨*proof*⟩

Parallel lifting example.

**lemma** ‹$\exists u.\ \forall x.\ \exists v.\ \forall y.\ \exists w.\ P(u,x,v,y,w)$›
⟨*proof*⟩

**lemma**
  **assumes** ‹$(\exists z.\ F(z)) \wedge B$›
  **shows** ‹$\exists z.\ F(z) \wedge B$›
⟨*proof*⟩

A bigger demonstration of quantifiers – not in the paper.

**lemma** ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
⟨*proof*⟩

**end**

# 6 First-Order Logic: PROLOG examples

**theory** *Prolog*
**imports** *FOL*
**begin**

**typedecl** *'a list*
**instance** *list* :: (‹*term*›) ‹*term*› ⟨*proof*⟩

**axiomatization**
  *Nil*   :: ‹*'a list*› **and**
  *Cons*  :: ‹[*'a*, *'a list*]=> *'a list*›   (**infixr** ‹:› *60*) **and**
  *app*   :: ‹[*'a list*, *'a list*, *'a list*] => *o*› **and**
  *rev*   :: ‹[*'a list*, *'a list*] => *o*›
**where**
  *appNil*: ‹*app*(*Nil*,*ys*,*ys*)› **and**
  *appCons*: ‹*app*(*xs*,*ys*,*zs*) ==> *app*(*x*:*xs*, *ys*, *x*:*zs*)› **and**
  *revNil*: ‹*rev*(*Nil*,*Nil*)› **and**
  *revCons*: ‹[| *rev*(*xs*,*ys*);  *app*(*ys*, *x*:*Nil*, *zs*) |] ==> *rev*(*x*:*xs*, *zs*)›

**schematic-goal** ‹*app*(*a*:*b*:*c*:*Nil*, *d*:*e*:*Nil*, *?x*)›
⟨*proof*⟩

**schematic-goal** ‹*app*(*?x*, *c*:*d*:*Nil*, *a*:*b*:*c*:*d*:*Nil*)›
⟨*proof*⟩

**schematic-goal** ‹*app*(*?x*, *?y*, *a*:*b*:*c*:*d*:*Nil*)›

⟨*proof*⟩

**lemmas** *rules* = *appNil appCons revNil revCons*

**schematic-goal** ‹$rev(a{:}b{:}c{:}d{:}Nil,\ ?x)$›
⟨*proof*⟩

**schematic-goal** ‹$rev(a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}Nil,\ ?w)$›
⟨*proof*⟩

**schematic-goal** ‹$rev(?x,\ a{:}b{:}c{:}Nil)$›
⟨*proof*⟩

⟨*ML*⟩

**schematic-goal** ‹$rev(?x,\ a{:}b{:}c{:}Nil)$›
⟨*proof*⟩

**schematic-goal** ‹$rev(a{:}?x{:}c{:}?y{:}Nil,\ d{:}?z{:}b{:}?u)$›
⟨*proof*⟩

**schematic-goal** ‹$rev(a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}o{:}p{:}Nil,\ ?w)$›
⟨*proof*⟩

**schematic-goal** ‹$a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}o{:}p{:}Nil = ?x \wedge app(?x,?x,?y) \wedge rev(?y,?w)$›
⟨*proof*⟩

**end**

# 7 Intuitionistic First-Order Logic

**theory** *Intuitionistic*
**imports** *IFOL*
**begin**

Metatheorem (for *propositional* formulae): $P$ is classically provable iff $\neg\neg P$ is intuitionistically provable. Therefore $\neg P$ is classically provable iff it is intuitionistically provable.

Proof: Let $Q$ be the conjunction of the propositions $A \vee \neg A$, one for each atom $A$ in $P$. Now $\neg\neg Q$ is intuitionistically provable because $\neg\neg(A \vee \neg A)$ is and because double-negation distributes over conjunction. If $P$ is provable classically, then clearly $Q \rightarrow P$ is provable intuitionistically, so $\neg\neg(Q \rightarrow P)$

is also provable intuitionistically. The latter is intuitionistically equivalent to $\neg\neg Q \rightarrow \neg\neg P$, hence to $\neg\neg P$, since $\neg\neg Q$ is intuitionistically provable. Finally, if $P$ is a negation then $\neg\neg P$ is intuitionstically equivalent to $P$. [Andy Pitts]

**lemma** ‹$\neg \neg (P \wedge Q) \longleftrightarrow \neg \neg P \wedge \neg \neg Q$›
⟨*proof*⟩

**lemma** ‹$\neg \neg ((\neg P \longrightarrow Q) \longrightarrow (\neg P \longrightarrow \neg Q) \longrightarrow P)$›
⟨*proof*⟩

Double-negation does NOT distribute over disjunction.

**lemma** ‹$\neg \neg (P \longrightarrow Q) \longleftrightarrow (\neg \neg P \longrightarrow \neg \neg Q)$›
⟨*proof*⟩

**lemma** ‹$\neg \neg \neg P \longleftrightarrow \neg P$›
⟨*proof*⟩

**lemma** ‹$\neg \neg ((P \longrightarrow Q \vee R) \longrightarrow (P \longrightarrow Q) \vee (P \longrightarrow R))$›
⟨*proof*⟩

**lemma** ‹$(P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P)$›
⟨*proof*⟩

**lemma** ‹$((P \longrightarrow (Q \vee (Q \longrightarrow R))) \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

**lemma**
‹$(((G \longrightarrow A) \longrightarrow J) \longrightarrow D \longrightarrow E) \longrightarrow (((H \longrightarrow B) \longrightarrow I) \longrightarrow C \longrightarrow J)$
$\longrightarrow (A \longrightarrow H) \longrightarrow F \longrightarrow G \longrightarrow (((C \longrightarrow B) \longrightarrow I) \longrightarrow D) \longrightarrow (A \longrightarrow C)$
$\longrightarrow (((F \longrightarrow A) \longrightarrow B) \longrightarrow I) \longrightarrow E$›
⟨*proof*⟩

Admissibility of the excluded middle for negated formulae

**lemma** ‹$(P \vee \neg P \longrightarrow \neg Q) \longrightarrow \neg Q$›
⟨*proof*⟩

The same in a more general form, no ex falso quodlibet

**lemma** ‹$(P \vee (P \longrightarrow R) \longrightarrow Q \longrightarrow R) \longrightarrow Q \longrightarrow R$›
⟨*proof*⟩

## 7.1 Lemmas for the propositional double-negation translation

**lemma** ‹$P \longrightarrow \neg \neg P$›
⟨*proof*⟩

**lemma** ‹$\neg \neg (\neg \neg P \longrightarrow P)$›
⟨*proof*⟩

**lemma** ‹$\neg\neg P \wedge \neg\neg(P \longrightarrow Q) \longrightarrow \neg\neg Q$›
⟨*proof*⟩

The following are classically but not constructively valid. The attempt to prove them terminates quickly!

**lemma** ‹$((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

**lemma** ‹$(P \wedge Q \longrightarrow R) \longrightarrow (P \longrightarrow R) \vee (Q \longrightarrow R)$›
⟨*proof*⟩

## 7.2 de Bruijn formulae

de Bruijn formula with three predicates

**lemma**
‹$((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R) \wedge$
$((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R) \wedge$
$((R \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R) \longrightarrow P \wedge Q \wedge R$›
⟨*proof*⟩

de Bruijn formula with five predicates

**lemma**
‹$((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((R \longleftrightarrow S) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((S \longleftrightarrow T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((T \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T$›
⟨*proof*⟩

Problems from of Sahlin, Franzen and Haridi, An Intuitionistic Predicate Logic Theorem Prover. J. Logic and Comp. 2 (5), October 1992, 619-656.

Problem 1.1

**lemma**
‹$(\forall x.\ \exists y.\ \forall z.\ p(x) \wedge q(y) \wedge r(z)) \longleftrightarrow$
$(\forall z.\ \exists y.\ \forall x.\ p(x) \wedge q(y) \wedge r(z))$›
⟨*proof*⟩

Problem 3.1

**lemma** ‹$\neg(\exists x.\ \forall y.\ mem(y,x) \longleftrightarrow \neg mem(x,x))$›
⟨*proof*⟩

Problem 4.1: hopeless!

**lemma**
‹$(\forall x.\ p(x) \longrightarrow p(h(x)) \vee p(g(x))) \wedge (\exists x.\ p(x)) \wedge (\forall x.\ \neg p(h(x)))$
$\longrightarrow (\exists x.\ p(g(g(g(g(g(x)))))))$›
⟨*proof*⟩

## 7.3 Intuitionistic FOL: propositional problems based on Pelletier.

¬¬1

**lemma** ‹$\neg\ \neg\ ((P \longrightarrow Q) \longleftrightarrow (\neg\ Q \longrightarrow \neg\ P))$›
⟨*proof*⟩

¬¬2

**lemma** ‹$\neg\ \neg\ (\neg\ \neg\ P \longleftrightarrow P)$›
⟨*proof*⟩

3

**lemma** ‹$\neg\ (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩

¬¬4

**lemma** ‹$\neg\ \neg\ ((\neg\ P \longrightarrow Q) \longleftrightarrow (\neg\ Q \longrightarrow P))$›
⟨*proof*⟩

¬¬5

**lemma** ‹$\neg\ \neg\ ((P \vee Q \longrightarrow P \vee R) \longrightarrow P \vee (Q \longrightarrow R))$›
⟨*proof*⟩

¬¬6

**lemma** ‹$\neg\ \neg\ (P \vee \neg\ P)$›
⟨*proof*⟩

¬¬7

**lemma** ‹$\neg\ \neg\ (P \vee \neg\ \neg\ \neg\ P)$›
⟨*proof*⟩

¬¬8. Peirce's law

**lemma** ‹$\neg\ \neg\ (((P \longrightarrow Q) \longrightarrow P) \longrightarrow P)$›
⟨*proof*⟩

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**lemma** ‹$((P \vee Q) \wedge (\neg\ P \vee Q) \wedge (P \vee \neg\ Q)) \longrightarrow \neg\ (\neg\ P \vee \neg\ Q)$›
⟨*proof*⟩

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**lemma** ‹$(Q \longrightarrow R) \longrightarrow (R \longrightarrow P \wedge Q) \longrightarrow (P \longrightarrow (Q \vee R)) \longrightarrow (P \longleftrightarrow Q)$›
⟨*proof*⟩

### 7.4 11. Proved in each direction (incorrectly, says Pelletier!!)

**lemma** ‹$P \longleftrightarrow P$›
  ⟨*proof*⟩

¬¬12. Dijkstra's law

**lemma** ‹$\neg \neg (((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R)))$›
  ⟨*proof*⟩

**lemma** ‹$((P \longleftrightarrow Q) \longleftrightarrow R) \longrightarrow \neg \neg (P \longleftrightarrow (Q \longleftrightarrow R))$›
  ⟨*proof*⟩

13. Distributive law

**lemma** ‹$P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R)$›
  ⟨*proof*⟩

¬¬14

**lemma** ‹$\neg \neg ((P \longleftrightarrow Q) \longleftrightarrow ((Q \vee \neg P) \wedge (\neg Q \vee P)))$›
  ⟨*proof*⟩

¬¬15

**lemma** ‹$\neg \neg ((P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q))$›
  ⟨*proof*⟩

¬¬16

**lemma** ‹$\neg \neg ((P \longrightarrow Q) \vee (Q \longrightarrow P))$›
  ⟨*proof*⟩

¬¬17

**lemma** ‹$\neg \neg (((P \wedge (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S)))$›
  ⟨*proof*⟩

Dijkstra's "Golden Rule"

**lemma** ‹$(P \wedge Q) \longleftrightarrow P \longleftrightarrow Q \longleftrightarrow (P \vee Q)$›
  ⟨*proof*⟩

## 8 Examples with quantifiers

### 8.1 The converse is classical in the following implications . . .

**lemma** ‹$(\exists x.\ P(x) \longrightarrow Q) \longrightarrow (\forall x.\ P(x)) \longrightarrow Q$›
  ⟨*proof*⟩

**lemma** ‹$((\forall x.\ P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x.\ P(x) \wedge \neg Q)$›
  ⟨*proof*⟩

**lemma** ‹$((\forall x.\ \neg P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x.\ \neg (P(x) \vee Q))$›

⟨*proof*⟩

**lemma** ‹$(\forall x.\ P(x)) \vee Q \longrightarrow (\forall x.\ P(x) \vee Q)$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ P \longrightarrow Q(x)) \longrightarrow (P \longrightarrow (\exists x.\ Q(x)))$›
⟨*proof*⟩

## 8.2 The following are not constructively valid!

The attempt to prove them terminates quickly!

**lemma** ‹$((\forall x.\ P(x)) \longrightarrow Q) \longrightarrow (\exists x.\ P(x) \longrightarrow Q)$›
⟨*proof*⟩

**lemma** ‹$(P \longrightarrow (\exists x.\ Q(x))) \longrightarrow (\exists x.\ P \longrightarrow Q(x))$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ P(x) \vee Q) \longrightarrow ((\forall x.\ P(x)) \vee Q)$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ \neg\neg P(x)) \longrightarrow \neg\neg(\forall x.\ P(x))$›
⟨*proof*⟩

Classically but not intuitionistically valid. Proved by a bug in 1986!

**lemma** ‹$\exists x.\ Q(x) \longrightarrow (\forall x.\ Q(x))$›
⟨*proof*⟩

## 8.3 Hard examples with quantifiers

The ones that have not been proved are not known to be valid! Some will require quantifier duplication – not currently available.

¬¬18

**lemma** ‹$\neg\neg(\exists y.\ \forall x.\ P(y) \longrightarrow P(x))$›
⟨*proof*⟩

¬¬19

**lemma** ‹$\neg\neg(\exists x.\ \forall y\ z.\ (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x)))$›
⟨*proof*⟩

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**lemma**
‹$(\forall x\ y.\ \exists z.\ \forall w.\ (P(x) \wedge Q(y) \longrightarrow R(z) \wedge S(w)))$
$\longrightarrow (\exists x\ y.\ P(x) \wedge Q(y)) \longrightarrow (\exists z.\ R(z))$›
⟨*proof*⟩

21

**lemma** ‹$(\exists x.\ P \longrightarrow Q(x)) \wedge (\exists x.\ Q(x) \longrightarrow P) \longrightarrow \neg\neg (\exists x.\ P \longleftrightarrow Q(x))$›
  ⟨*proof*⟩

22

**lemma** ‹$(\forall x.\ P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x.\ Q(x)))$›
  ⟨*proof*⟩

¬¬23

**lemma** ‹$\neg\neg ((\forall x.\ P \vee Q(x)) \longleftrightarrow (P \vee (\forall x.\ Q(x))))$›
  ⟨*proof*⟩

24

**lemma**
  ‹$\neg (\exists x.\ S(x) \wedge Q(x)) \wedge (\forall x.\ P(x) \longrightarrow Q(x) \vee R(x)) \wedge$
    $(\neg (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))) \wedge (\forall x.\ Q(x) \vee R(x) \longrightarrow S(x))$
    $\longrightarrow \neg\neg (\exists x.\ P(x) \wedge R(x))$›

Not clear why *fast-tac*, *best-tac*, *ASTAR* and *ITER-DEEPEN* all take forever.

  ⟨*proof*⟩

25

**lemma**
  ‹$(\exists x.\ P(x)) \wedge$
      $(\forall x.\ L(x) \longrightarrow \neg (M(x) \wedge R(x))) \wedge$
      $(\forall x.\ P(x) \longrightarrow (M(x) \wedge L(x))) \wedge$
      $((\forall x.\ P(x) \longrightarrow Q(x)) \vee (\exists x.\ P(x) \wedge R(x)))$
    $\longrightarrow (\exists x.\ Q(x) \wedge P(x))$›
  ⟨*proof*⟩

¬¬26

**lemma**
  ‹$(\neg\neg (\exists x.\ p(x)) \longleftrightarrow \neg\neg (\exists x.\ q(x))) \wedge$
    $(\forall x.\ \forall y.\ p(x) \wedge q(y) \longrightarrow (r(x) \longleftrightarrow s(y)))$
  $\longrightarrow ((\forall x.\ p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x.\ q(x) \longrightarrow s(x)))$›
  ⟨*proof*⟩

27

**lemma**
  ‹$(\exists x.\ P(x) \wedge \neg Q(x)) \wedge$
    $(\forall x.\ P(x) \longrightarrow R(x)) \wedge$
    $(\forall x.\ M(x) \wedge L(x) \longrightarrow P(x)) \wedge$
    $((\exists x.\ R(x) \wedge \neg Q(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow \neg R(x)))$
  $\longrightarrow (\forall x.\ M(x) \longrightarrow \neg L(x))$›
  ⟨*proof*⟩

¬¬28. AMENDED

**lemma**

‹$(\forall x.\ P(x) \longrightarrow (\forall x.\ Q(x))) \wedge$
$(\neg\ \neg\ (\forall x.\ Q(x) \vee R(x)) \longrightarrow (\exists x.\ Q(x) \wedge S(x))) \wedge$
$(\neg\ \neg\ (\exists x.\ S(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow M(x)))$
$\longrightarrow (\forall x.\ P(x) \wedge L(x) \longrightarrow M(x))$›
⟨*proof*⟩

29. Essentially the same as Principia Mathematica *11.71

**lemma**
‹$(\exists x.\ P(x)) \wedge (\exists y.\ Q(y))$
$\longrightarrow ((\forall x.\ P(x) \longrightarrow R(x)) \wedge (\forall y.\ Q(y) \longrightarrow S(y)) \longleftrightarrow$
$(\forall x\ y.\ P(x) \wedge Q(y) \longrightarrow R(x) \wedge S(y)))$›
⟨*proof*⟩

¬¬30

**lemma**
‹$(\forall x.\ (P(x) \vee Q(x)) \longrightarrow \neg\ R(x)) \wedge$
$(\forall x.\ (Q(x) \longrightarrow \neg\ S(x)) \longrightarrow P(x) \wedge R(x))$
$\longrightarrow (\forall x.\ \neg\ \neg\ S(x))$›
⟨*proof*⟩

31

**lemma**
‹$\neg\ (\exists x.\ P(x) \wedge (Q(x) \vee R(x))) \wedge$
$(\exists x.\ L(x) \wedge P(x)) \wedge$
$(\forall x.\ \neg\ R(x) \longrightarrow M(x))$
$\longrightarrow (\exists x.\ L(x) \wedge M(x))$›
⟨*proof*⟩

32

**lemma**
‹$(\forall x.\ P(x) \wedge (Q(x) \vee R(x)) \longrightarrow S(x)) \wedge$
$(\forall x.\ S(x) \wedge R(x) \longrightarrow L(x)) \wedge$
$(\forall x.\ M(x) \longrightarrow R(x))$
$\longrightarrow (\forall x.\ P(x) \wedge M(x) \longrightarrow L(x))$›
⟨*proof*⟩

¬¬33

**lemma**
‹$(\forall x.\ \neg\ \neg\ (P(a) \wedge (P(x) \longrightarrow P(b)) \longrightarrow P(c))) \longleftrightarrow$
$(\forall x.\ \neg\ \neg\ ((\neg\ P(a) \vee P(x) \vee P(c)) \wedge (\neg\ P(a) \vee \neg\ P(b) \vee P(c))))$›
⟨*proof*⟩

36

**lemma**
‹$(\forall x.\ \exists y.\ J(x,y)) \wedge$
$(\forall x.\ \exists y.\ G(x,y)) \wedge$
$(\forall x\ y.\ J(x,y) \vee G(x,y) \longrightarrow (\forall z.\ J(y,z) \vee G(y,z) \longrightarrow H(x,z)))$
$\longrightarrow (\forall x.\ \exists y.\ H(x,y))$›

$\langle proof \rangle$

37

**lemma**
‹$(\forall z.\ \exists w.\ \forall x.\ \exists y.$
$\neg\neg (P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u.\ Q(u,w)))) \wedge$
$(\forall x\ z.\ \neg P(x,z) \longrightarrow (\exists y.\ Q(y,z))) \wedge$
$(\neg\neg (\exists x\ y.\ Q(x,y)) \longrightarrow (\forall x.\ R(x,x)))$
$\longrightarrow \neg\neg (\forall x.\ \exists y.\ R(x,y))$›
$\langle proof \rangle$

39

**lemma** ‹$\neg (\exists x.\ \forall y.\ F(y,x) \longleftrightarrow \neg F(y,y))$›
$\langle proof \rangle$

40. AMENDED

**lemma**
‹$(\exists y.\ \forall x.\ F(x,y) \longleftrightarrow F(x,x)) \longrightarrow$
$\neg (\forall x.\ \exists y.\ \forall z.\ F(z,y) \longleftrightarrow \neg F(z,x))$›
$\langle proof \rangle$

44

**lemma**
‹$(\forall x.\ f(x) \longrightarrow$
$(\exists y.\ g(y) \wedge h(x,y) \wedge (\exists y.\ g(y) \wedge \neg h(x,y)))) \wedge$
$(\exists x.\ j(x) \wedge (\forall y.\ g(y) \longrightarrow h(x,y)))$
$\longrightarrow (\exists x.\ j(x) \wedge \neg f(x))$›
$\langle proof \rangle$

48

**lemma** ‹$(a = b \vee c = d) \wedge (a = c \vee b = d) \longrightarrow a = d \vee b = c$›
$\langle proof \rangle$

51

**lemma**
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
$(\exists z.\ \forall x.\ \exists w.\ (\forall y.\ P(x,y) \longleftrightarrow y = w) \longleftrightarrow x = z)$›
$\langle proof \rangle$

52

Almost the same as 51.

**lemma**
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
$(\exists w.\ \forall y.\ \exists z.\ (\forall x.\ P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w)$›
$\langle proof \rangle$

56

**lemma** ‹$(\forall x.\ (\exists y.\ P(y) \land x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x.\ P(x) \longrightarrow P(f(x)))$›
⟨*proof*⟩

57

**lemma**
‹$P(f(a,b),\ f(b,c)) \land P(f(b,c),\ f(a,c)) \land$
$(\forall x\ y\ z.\ P(x,y) \land P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b),\ f(a,c))$›
⟨*proof*⟩

60

**lemma** ‹$\forall x.\ P(x,f(x)) \longleftrightarrow (\exists y.\ (\forall z.\ P(z,y) \longrightarrow P(z,f(x))) \land P(x,y))$›
⟨*proof*⟩

**end**

# 9 First-Order Logic: propositional examples (intuitionistic version)

**theory** *Propositional-Int*
**imports** *IFOL*
**begin**

commutative laws of ∧ and ∨

**lemma** ‹$P \land Q \longrightarrow Q \land P$›
⟨*proof*⟩

**lemma** ‹$P \lor Q \longrightarrow Q \lor P$›
⟨*proof*⟩

associative laws of ∧ and ∨

**lemma** ‹$(P \land Q) \land R \longrightarrow P \land (Q \land R)$›
⟨*proof*⟩

**lemma** ‹$(P \lor Q) \lor R \longrightarrow P \lor (Q \lor R)$›
⟨*proof*⟩

distributive laws of ∧ and ∨

**lemma** ‹$(P \land Q) \lor R \longrightarrow (P \lor R) \land (Q \lor R)$›
⟨*proof*⟩

**lemma** ‹$(P \lor R) \land (Q \lor R) \longrightarrow (P \land Q) \lor R$›
⟨*proof*⟩

**lemma** ‹$(P \lor Q) \land R \longrightarrow (P \land R) \lor (Q \land R)$›
⟨*proof*⟩

**lemma** ‹$(P \land R) \lor (Q \land R) \longrightarrow (P \lor Q) \land R$›

⟨*proof*⟩

Laws involving implication

**lemma** ‹$(P \longrightarrow R) \wedge (Q \longrightarrow R) \longleftrightarrow (P \vee Q \longrightarrow R)$›
⟨*proof*⟩

**lemma** ‹$(P \wedge Q \longrightarrow R) \longleftrightarrow (P \longrightarrow (Q \longrightarrow R))$›
⟨*proof*⟩

**lemma** ‹$((P \longrightarrow R) \longrightarrow R) \longrightarrow ((Q \longrightarrow R) \longrightarrow R) \longrightarrow (P \wedge Q \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

**lemma** ‹$\neg (P \longrightarrow R) \longrightarrow \neg (Q \longrightarrow R) \longrightarrow \neg (P \wedge Q \longrightarrow R)$›
⟨*proof*⟩

**lemma** ‹$(P \longrightarrow Q \wedge R) \longleftrightarrow (P \longrightarrow Q) \wedge (P \longrightarrow R)$›
⟨*proof*⟩

Propositions-as-types

**lemma** ‹$P \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩
**lemma** ‹$(P \longrightarrow Q \longrightarrow R) \longrightarrow (P \longrightarrow Q) \longrightarrow (P \longrightarrow R)$›
⟨*proof*⟩
**lemma** ‹$(P \longrightarrow Q) \vee (P \longrightarrow R) \longrightarrow (P \longrightarrow Q \vee R)$›
⟨*proof*⟩

**lemma** ‹$(P \longrightarrow Q) \longrightarrow (\neg Q \longrightarrow \neg P)$›
⟨*proof*⟩

Schwichtenberg's examples (via T. Nipkow)

**lemma** *stab-imp*: ‹$(((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow R) \longrightarrow P \longrightarrow Q$›
⟨*proof*⟩

**lemma** *stab-to-peirce*:
‹$(((P \longrightarrow R) \longrightarrow R) \longrightarrow P) \longrightarrow (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q)$
$\longrightarrow ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

**lemma** *peirce-imp1*:
‹$(((Q \longrightarrow R) \longrightarrow Q) \longrightarrow Q)$
$\longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow P \longrightarrow Q) \longrightarrow P \longrightarrow Q$›
⟨*proof*⟩

**lemma** *peirce-imp2*: ‹$(((P \longrightarrow R) \longrightarrow P) \longrightarrow P) \longrightarrow ((P \longrightarrow Q \longrightarrow R) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

**lemma** *mints*: ‹$((((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \longrightarrow Q) \longrightarrow Q$›

⟨*proof*⟩

**lemma** *mints-solovev*: ‹$(P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

**lemma** *tatsuta*:
‹$(((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9) \longrightarrow P5$›
⟨*proof*⟩

**lemma** *tatsuta1*:
‹$(((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9)$
$\longrightarrow (((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7 \longrightarrow P5$›
⟨*proof*⟩

**end**

# 10 First-Order Logic: quantifier examples (intuitionistic version)

**theory** *Quantifiers-Int*
**imports** *IFOL*
**begin**

**lemma** ‹$(\forall x\ y.\ P(x,y)) \longrightarrow (\forall y\ x.\ P(x,y))$›
⟨*proof*⟩

**lemma** ‹$(\exists x\ y.\ P(x,y)) \longrightarrow (\exists y\ x.\ P(x,y))$›
⟨*proof*⟩
**lemma** ‹$(\forall x.\ P(x)) \vee (\forall x.\ Q(x)) \longrightarrow (\forall x.\ P(x) \vee Q(x))$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x.\ Q(x)))$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ P(x) \longrightarrow Q) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow Q)$›
⟨*proof*⟩

Some harder ones

**lemma** ‹$(\exists x.\ P(x) \vee Q(x)) \longleftrightarrow (\exists x.\ P(x)) \vee (\exists x.\ Q(x))$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ P(x) \wedge Q(x)) \longrightarrow (\exists x.\ P(x)) \wedge (\exists x.\ Q(x))$›
⟨*proof*⟩

Basic test of quantifier reasoning

**lemma** ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

The following should fail, as they are false!

**lemma** ‹$(\forall x.\ \exists y.\ Q(x,y)) \longrightarrow (\exists y.\ \forall x.\ Q(x,y))$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ Q(x)) \longrightarrow (\forall x.\ Q(x))$›
⟨*proof*⟩

**schematic-goal** ‹$P(?a) \longrightarrow (\forall x.\ P(x))$›
⟨*proof*⟩

**schematic-goal** ‹$(P(?a) \longrightarrow (\forall x.\ Q(x))) \longrightarrow (\forall x.\ P(x) \longrightarrow Q(x))$›
⟨*proof*⟩

Back to things that are provable . . .

**lemma** ‹$(\forall x.\ P(x) \longrightarrow Q(x)) \wedge (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩
**lemma** ‹$(P \longrightarrow (\exists x.\ Q(x))) \wedge P \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

**schematic-goal** ‹$(\forall x.\ P(x) \longrightarrow Q(f(x))) \wedge (\forall x.\ Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a)$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

Some slow ones

**lemma** ‹$(\forall x\ y.\ P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow (\forall y.\ Q(y)))$›
⟨*proof*⟩

**lemma** ‹$(\exists x\ y.\ P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x.\ P(x) \wedge (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

**lemma** ‹$(\exists y.\ \forall x.\ P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x.\ P(x) \longrightarrow (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

**end**

# 11 Classical Predicate Calculus Problems

**theory** *Classical*
**imports** *FOL*
**begin**

**lemma** ‹$(P \longrightarrow Q \vee R) \longrightarrow (P \longrightarrow Q) \vee (P \longrightarrow R)$›
⟨*proof*⟩

### 11.0.1 If and only if

**lemma** ‹$(P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P)$›
⟨*proof*⟩

**lemma** ‹$\neg (P \longleftrightarrow \neg P)$›
⟨*proof*⟩

## 11.1 Pelletier's examples

Sample problems from

- F. J. Pelletier, Seventy-Five Problems for Testing Automatic Theorem Provers, J. Automated Reasoning 2 (1986), 191-216. Errata, JAR 4 (1988), 236-236.

The hardest problems – judging by experience with several theorem provers, including matrix ones – are 34 and 43.

1

**lemma** ‹$(P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow \neg P)$›
⟨*proof*⟩

2

**lemma** ‹$\neg \neg P \longleftrightarrow P$›
⟨*proof*⟩

3

**lemma** ‹$\neg (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩

4

**lemma** ‹$(\neg P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow P)$›
⟨*proof*⟩

5

**lemma** ‹$((P \vee Q) \longrightarrow (P \vee R)) \longrightarrow (P \vee (Q \longrightarrow R))$›
⟨*proof*⟩

6

**lemma** ‹$P \lor \neg P$›
⟨*proof*⟩

7

**lemma** ‹$P \lor \neg \neg \neg P$›
⟨*proof*⟩

8. Peirce's law

**lemma** ‹$((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

9

**lemma** ‹$((P \lor Q) \land (\neg P \lor Q) \land (P \lor \neg Q)) \longrightarrow \neg (\neg P \lor \neg Q)$›
⟨*proof*⟩

10

**lemma** ‹$(Q \longrightarrow R) \land (R \longrightarrow P \land Q) \land (P \longrightarrow Q \lor R) \longrightarrow (P \longleftrightarrow Q)$›
⟨*proof*⟩

11. Proved in each direction (incorrectly, says Pelletier!!)

**lemma** ‹$P \longleftrightarrow P$›
⟨*proof*⟩

12. "Dijkstra's law"

**lemma** ‹$((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R))$›
⟨*proof*⟩

13. Distributive law

**lemma** ‹$P \lor (Q \land R) \longleftrightarrow (P \lor Q) \land (P \lor R)$›
⟨*proof*⟩

14

**lemma** ‹$(P \longleftrightarrow Q) \longleftrightarrow ((Q \lor \neg P) \land (\neg Q \lor P))$›
⟨*proof*⟩

15

**lemma** ‹$(P \longrightarrow Q) \longleftrightarrow (\neg P \lor Q)$›
⟨*proof*⟩

16

**lemma** ‹$(P \longrightarrow Q) \lor (Q \longrightarrow P)$›
⟨*proof*⟩

17

**lemma** ‹$((P \land (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \lor Q \lor S) \land (\neg P \lor \neg R \lor S))$›
⟨*proof*⟩

### 11.2 Classical Logic: examples with quantifiers

**lemma** ‹$(\forall x.\ P(x) \wedge Q(x)) \longleftrightarrow (\forall x.\ P(x)) \wedge (\forall x.\ Q(x))$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\exists x.\ Q(x)))$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ P(x) \longrightarrow Q) \longleftrightarrow (\forall x.\ P(x)) \longrightarrow Q$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ P(x)) \vee Q \longleftrightarrow (\forall x.\ P(x) \vee Q)$›
⟨*proof*⟩

Discussed in Avron, Gentzen-Type Systems, Resolution and Tableaux, JAR 10 (265-281), 1993. Proof is trivial!

**lemma** ‹$\neg ((\exists x.\ \neg P(x)) \wedge ((\exists x.\ P(x)) \vee (\exists x.\ P(x) \wedge Q(x))) \wedge \neg (\exists x.\ P(x)))$›
⟨*proof*⟩

### 11.3 Problems requiring quantifier duplication

Theorem B of Peter Andrews, Theorem Proving via General Matings, JACM 28 (1981).

**lemma** ‹$(\exists x.\ \forall y.\ P(x) \longleftrightarrow P(y)) \longrightarrow ((\exists x.\ P(x)) \longleftrightarrow (\forall y.\ P(y)))$›
⟨*proof*⟩

Needs multiple instantiation of ALL.

**lemma** ‹$(\forall x.\ P(x) \longrightarrow P(f(x))) \wedge P(d) \longrightarrow P(f(f(f(d))))$›
⟨*proof*⟩

Needs double instantiation of the quantifier

**lemma** ‹$\exists x.\ P(x) \longrightarrow P(a) \wedge P(b)$›
⟨*proof*⟩

**lemma** ‹$\exists z.\ P(z) \longrightarrow (\forall x.\ P(x))$›
⟨*proof*⟩

**lemma** ‹$\exists x.\ (\exists y.\ P(y)) \longrightarrow P(x)$›
⟨*proof*⟩

V. Lifschitz, What Is the Inverse Method?, JAR 5 (1989), 1–23. NOT PROVED.

**lemma**
‹$\exists x\ x'.\ \forall y.\ \exists z\ z'.$
$(\neg P(y,y) \vee P(x,x) \vee \neg S(z,x)) \wedge$
$(S(x,y) \vee \neg S(y,z) \vee Q(z',z')) \wedge$
$(Q(x',y) \vee \neg Q(y,z') \vee S(x',x'))$›
⟨*proof*⟩

## 11.4 Hard examples with quantifiers

18

**lemma** ‹$\exists y.\ \forall x.\ P(y) \longrightarrow P(x)$›
⟨*proof*⟩

19

**lemma** ‹$\exists x.\ \forall y\ z.\ (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x))$›
⟨*proof*⟩

20

**lemma** ‹$(\forall x\ y.\ \exists z.\ \forall w.\ (P(x) \wedge Q(y) \longrightarrow R(z) \wedge S(w)))$
$\longrightarrow (\exists x\ y.\ P(x) \wedge Q(y)) \longrightarrow (\exists z.\ R(z))$›
⟨*proof*⟩

21

**lemma** ‹$(\exists x.\ P \longrightarrow Q(x)) \wedge (\exists x.\ Q(x) \longrightarrow P) \longrightarrow (\exists x.\ P \longleftrightarrow Q(x))$›
⟨*proof*⟩

22

**lemma** ‹$(\forall x.\ P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x.\ Q(x)))$›
⟨*proof*⟩

23

**lemma** ‹$(\forall x.\ P \vee Q(x)) \longleftrightarrow (P \vee (\forall x.\ Q(x)))$›
⟨*proof*⟩

24

**lemma**
‹$\neg\ (\exists x.\ S(x) \wedge Q(x)) \wedge (\forall x.\ P(x) \longrightarrow Q(x) \vee R(x)) \wedge$
$(\neg\ (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))) \wedge (\forall x.\ Q(x) \vee R(x) \longrightarrow S(x))$
$\longrightarrow (\exists x.\ P(x) \wedge R(x))$›
⟨*proof*⟩

25

**lemma**
‹$(\exists x.\ P(x)) \wedge$
$(\forall x.\ L(x) \longrightarrow \neg\ (M(x) \wedge R(x))) \wedge$
$(\forall x.\ P(x) \longrightarrow (M(x) \wedge L(x))) \wedge$
$((\forall x.\ P(x) \longrightarrow Q(x)) \vee (\exists x.\ P(x) \wedge R(x)))$
$\longrightarrow (\exists x.\ Q(x) \wedge P(x))$›
⟨*proof*⟩

26

**lemma**
‹$((\exists x.\ p(x)) \longleftrightarrow (\exists x.\ q(x))) \wedge$
$(\forall x.\ \forall y.\ p(x) \wedge q(y) \longrightarrow (r(x) \longleftrightarrow s(y)))$

$\longrightarrow ((\forall x.\ p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x.\ q(x) \longrightarrow s(x)))$›
⟨*proof*⟩

27

**lemma**
‹$(\exists x.\ P(x) \wedge \neg\ Q(x)) \wedge$
$(\forall x.\ P(x) \longrightarrow R(x)) \wedge$
$(\forall x.\ M(x) \wedge L(x) \longrightarrow P(x)) \wedge$
$((\exists x.\ R(x) \wedge \neg\ Q(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow \neg\ R(x)))$
$\longrightarrow (\forall x.\ M(x) \longrightarrow \neg\ L(x))$›
⟨*proof*⟩

28. AMENDED

**lemma**
‹$(\forall x.\ P(x) \longrightarrow (\forall x.\ Q(x))) \wedge$
$((\forall x.\ Q(x) \vee R(x)) \longrightarrow (\exists x.\ Q(x) \wedge S(x))) \wedge$
$((\exists x.\ S(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow M(x)))$
$\longrightarrow (\forall x.\ P(x) \wedge L(x) \longrightarrow M(x))$›
⟨*proof*⟩

29. Essentially the same as Principia Mathematica *11.71

**lemma**
‹$(\exists x.\ P(x)) \wedge (\exists y.\ Q(y))$
$\longrightarrow ((\forall x.\ P(x) \longrightarrow R(x)) \wedge (\forall y.\ Q(y) \longrightarrow S(y)) \longleftrightarrow$
$(\forall x\ y.\ P(x) \wedge Q(y) \longrightarrow R(x) \wedge S(y)))$›
⟨*proof*⟩

30

**lemma**
‹$(\forall x.\ P(x) \vee Q(x) \longrightarrow \neg\ R(x)) \wedge$
$(\forall x.\ (Q(x) \longrightarrow \neg\ S(x)) \longrightarrow P(x) \wedge R(x))$
$\longrightarrow (\forall x.\ S(x))$›
⟨*proof*⟩

31

**lemma**
‹$\neg\ (\exists x.\ P(x) \wedge (Q(x) \vee R(x))) \wedge$
$(\exists x.\ L(x) \wedge P(x)) \wedge$
$(\forall x.\ \neg\ R(x) \longrightarrow M(x))$
$\longrightarrow (\exists x.\ L(x) \wedge M(x))$›
⟨*proof*⟩

32

**lemma**
‹$(\forall x.\ P(x) \wedge (Q(x) \vee R(x)) \longrightarrow S(x)) \wedge$
$(\forall x.\ S(x) \wedge R(x) \longrightarrow L(x)) \wedge$
$(\forall x.\ M(x) \longrightarrow R(x))$
$\longrightarrow (\forall x.\ P(x) \wedge M(x) \longrightarrow L(x))$›

⟨*proof*⟩

33

**lemma**
‹$(\forall x.\ P(a) \wedge (P(x) \longrightarrow P(b)) \longrightarrow P(c)) \longleftrightarrow$
$(\forall x.\ (\neg P(a) \vee P(x) \vee P(c)) \wedge (\neg P(a) \vee \neg P(b) \vee P(c)))$›
⟨*proof*⟩

34. AMENDED (TWICE!!). Andrews's challenge.

**lemma**
‹$((\exists x.\ \forall y.\ p(x) \longleftrightarrow p(y)) \longleftrightarrow ((\exists x.\ q(x)) \longleftrightarrow (\forall y.\ p(y)))) \longleftrightarrow$
$((\exists x.\ \forall y.\ q(x) \longleftrightarrow q(y)) \longleftrightarrow ((\exists x.\ p(x)) \longleftrightarrow (\forall y.\ q(y))))$›
⟨*proof*⟩

35

**lemma** ‹$\exists x\ y.\ P(x,y) \longrightarrow (\forall u\ v.\ P(u,v))$›
⟨*proof*⟩

36

**lemma**
‹$(\forall x.\ \exists y.\ J(x,y)) \wedge$
$(\forall x.\ \exists y.\ G(x,y)) \wedge$
$(\forall x\ y.\ J(x,y) \vee G(x,y) \longrightarrow (\forall z.\ J(y,z) \vee G(y,z) \longrightarrow H(x,z)))$
$\longrightarrow (\forall x.\ \exists y.\ H(x,y))$›
⟨*proof*⟩

37

**lemma**
‹$(\forall z.\ \exists w.\ \forall x.\ \exists y.$
$(P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u.\ Q(u,w)))) \wedge$
$(\forall x\ z.\ \neg P(x,z) \longrightarrow (\exists y.\ Q(y,z))) \wedge$
$((\exists x\ y.\ Q(x,y)) \longrightarrow (\forall x.\ R(x,x)))$
$\longrightarrow (\forall x.\ \exists y.\ R(x,y))$›
⟨*proof*⟩

38

**lemma**
‹$(\forall x.\ p(a) \wedge (p(x) \longrightarrow (\exists y.\ p(y) \wedge r(x,y))) \longrightarrow$
$(\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z))) \longleftrightarrow$
$(\forall x.\ (\neg p(a) \vee p(x) \vee (\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z))) \wedge$
$(\neg p(a) \vee \neg (\exists y.\ p(y) \wedge r(x,y)) \vee$
$(\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z))))$›
⟨*proof*⟩

39

**lemma** ‹$\neg (\exists x.\ \forall y.\ F(y,x) \longleftrightarrow \neg F(y,y))$›
⟨*proof*⟩

40. AMENDED

**lemma**
‹$(\exists y. \forall x. F(x,y) \longleftrightarrow F(x,x)) \longrightarrow$
$\neg (\forall x. \exists y. \forall z. F(z,y) \longleftrightarrow \neg F(z,x))$›
⟨*proof*⟩

41

**lemma**
‹$(\forall z. \exists y. \forall x. f(x,y) \longleftrightarrow f(x,z) \wedge \neg f(x,x))$
$\longrightarrow \neg (\exists z. \forall x. f(x,z))$›
⟨*proof*⟩

42

**lemma** ‹$\neg (\exists y. \forall x. p(x,y) \longleftrightarrow \neg (\exists z. p(x,z) \wedge p(z,x)))$›
⟨*proof*⟩

43

**lemma**
‹$(\forall x. \forall y. q(x,y) \longleftrightarrow (\forall z. p(z,x) \longleftrightarrow p(z,y)))$
$\longrightarrow (\forall x. \forall y. q(x,y) \longleftrightarrow q(y,x))$›
⟨*proof*⟩

Other proofs: Can use *auto*, which cheats by using rewriting! *Deepen-tac* alone requires 253 secs. Or *by* (*mini-tac 1 THEN Deepen-tac 5 1*).

44

**lemma**
‹$(\forall x. f(x) \longrightarrow (\exists y. g(y) \wedge h(x,y) \wedge (\exists y. g(y) \wedge \neg h(x,y)))) \wedge$
$(\exists x. j(x) \wedge (\forall y. g(y) \longrightarrow h(x,y)))$
$\longrightarrow (\exists x. j(x) \wedge \neg f(x))$›
⟨*proof*⟩

45

**lemma**
‹$(\forall x. f(x) \wedge (\forall y. g(y) \wedge h(x,y) \longrightarrow j(x,y))$
$\longrightarrow (\forall y. g(y) \wedge h(x,y) \longrightarrow k(y))) \wedge$
$\neg (\exists y. l(y) \wedge k(y)) \wedge$
$(\exists x. f(x) \wedge (\forall y. h(x,y) \longrightarrow l(y)) \wedge (\forall y. g(y) \wedge h(x,y) \longrightarrow j(x,y)))$
$\longrightarrow (\exists x. f(x) \wedge \neg (\exists y. g(y) \wedge h(x,y)))$›
⟨*proof*⟩

46

**lemma**
‹$(\forall x. f(x) \wedge (\forall y. f(y) \wedge h(y,x) \longrightarrow g(y)) \longrightarrow g(x)) \wedge$
$((\exists x. f(x) \wedge \neg g(x)) \longrightarrow$
$(\exists x. f(x) \wedge \neg g(x) \wedge (\forall y. f(y) \wedge \neg g(y) \longrightarrow j(x,y)))) \wedge$
$(\forall x\ y. f(x) \wedge f(y) \wedge h(x,y) \longrightarrow \neg j(y,x))$
$\longrightarrow (\forall x. f(x) \longrightarrow g(x))$›
⟨*proof*⟩

## 11.5 Problems (mainly) involving equality or functions

48

**lemma** ‹$(a = b \lor c = d) \land (a = c \lor b = d) \longrightarrow a = d \lor b = c$›
⟨*proof*⟩

49. NOT PROVED AUTOMATICALLY. Hard because it involves substitution for Vars; the type constraint ensures that x,y,z have the same type as a,b,u.

**lemma**
‹$(\exists x\ y{::}'a.\ \forall z.\ z = x \lor z = y) \land P(a) \land P(b) \land a \neq b \longrightarrow (\forall u{::}'a.\ P(u))$›
⟨*proof*⟩

50. (What has this to do with equality?)

**lemma** ‹$(\forall x.\ P(a,x) \lor (\forall y.\ P(x,y))) \longrightarrow (\exists x.\ \forall y.\ P(x,y))$›
⟨*proof*⟩

51

**lemma**
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \land y = w)) \longrightarrow$
$(\exists z.\ \forall x.\ \exists w.\ (\forall y.\ P(x,y) \longleftrightarrow y{=}w) \longleftrightarrow x = z)$›
⟨*proof*⟩

52

Almost the same as 51.

**lemma**
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \land y = w)) \longrightarrow$
$(\exists w.\ \forall y.\ \exists z.\ (\forall x.\ P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w)$›
⟨*proof*⟩

55

Non-equational version, from Manthey and Bry, CADE-9 (Springer, 1988). fast DISCOVERS who killed Agatha.

**schematic-goal**
‹$lives(agatha) \land lives(butler) \land lives(charles) \land$
$(killed(agatha,agatha) \lor killed(butler,agatha) \lor killed(charles,agatha)) \land$
$(\forall x\ y.\ killed(x,y) \longrightarrow hates(x,y) \land \neg\ richer(x,y)) \land$
$(\forall x.\ hates(agatha,x) \longrightarrow \neg\ hates(charles,x)) \land$
$(hates(agatha,agatha) \land hates(agatha,charles)) \land$
$(\forall x.\ lives(x) \land \neg\ richer(x,agatha) \longrightarrow hates(butler,x)) \land$
$(\forall x.\ hates(agatha,x) \longrightarrow hates(butler,x)) \land$
$(\forall x.\ \neg\ hates(x,agatha) \lor \neg\ hates(x,butler) \lor \neg\ hates(x,charles)) \longrightarrow$
$killed(?who,agatha)$›
⟨*proof*⟩

56

**lemma** ‹$(\forall x.\ (\exists y.\ P(y) \wedge x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x.\ P(x) \longrightarrow P(f(x)))$›
⟨*proof*⟩

57

**lemma**
‹$P(f(a,b),\ f(b,c)) \wedge P(f(b,c),\ f(a,c)) \wedge$
$(\forall x\ y\ z.\ P(x,y) \wedge P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b),\ f(a,c))$›
⟨*proof*⟩

58 NOT PROVED AUTOMATICALLY

**lemma** ‹$(\forall x\ y.\ f(x) = g(y)) \longrightarrow (\forall x\ y.\ f(f(x)) = f(g(y)))$›
⟨*proof*⟩

59

**lemma** ‹$(\forall x.\ P(x) \longleftrightarrow \neg\ P(f(x))) \longrightarrow (\exists x.\ P(x) \wedge \neg\ P(f(x)))$›
⟨*proof*⟩

60

**lemma** ‹$\forall x.\ P(x,f(x)) \longleftrightarrow (\exists y.\ (\forall z.\ P(z,y) \longrightarrow P(z,f(x))) \wedge P(x,y))$›
⟨*proof*⟩

62 as corrected in JAR 18 (1997), page 135

**lemma**
‹$(\forall x.\ p(a) \wedge (p(x) \longrightarrow p(f(x))) \longrightarrow p(f(f(x)))) \longleftrightarrow$
$(\forall x.\ (\neg\ p(a) \vee p(x) \vee p(f(f(x)))) \wedge$
$(\neg\ p(a) \vee \neg\ p(f(x)) \vee p(f(f(x)))))$›
⟨*proof*⟩

From Davis, Obvious Logical Inferences, IJCAI-81, 530-531 fast indeed copes!

**lemma**
‹$(\forall x.\ F(x) \wedge \neg\ G(x) \longrightarrow (\exists y.\ H(x,y) \wedge J(y))) \wedge$
$(\exists x.\ K(x) \wedge F(x) \wedge (\forall y.\ H(x,y) \longrightarrow K(y))) \wedge$
$(\forall x.\ K(x) \longrightarrow \neg\ G(x)) \longrightarrow (\exists x.\ K(x) \wedge J(x))$›
⟨*proof*⟩

From Rudnicki, Obvious Inferences, JAR 3 (1987), 383-393. It does seem obvious!

**lemma**
‹$(\forall x.\ F(x) \wedge \neg\ G(x) \longrightarrow (\exists y.\ H(x,y) \wedge J(y))) \wedge$
$(\exists x.\ K(x) \wedge F(x) \wedge (\forall y.\ H(x,y) \longrightarrow K(y))) \wedge$
$(\forall x.\ K(x) \longrightarrow \neg\ G(x)) \longrightarrow (\exists x.\ K(x) \longrightarrow \neg\ G(x))$›
⟨*proof*⟩

Halting problem: Formulation of Li Dafa (AAR Newsletter 27, Oct 1994.) author U. Egly.

**lemma**
‹$((\exists x.\ A(x) \wedge (\forall y.\ C(y) \longrightarrow (\forall z.\ D(x,y,z)))) \longrightarrow$

$(\exists\, w.\ C(w) \land (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(w,y,z)))))$
$\land$
$(\forall\, w.\ C(w) \land (\forall\, u.\ C(u) \longrightarrow (\forall\, v.\ D(w,u,v))) \longrightarrow$
$(\forall\, y\ z.$
$(C(y) \land P(y,z) \longrightarrow Q(w,y,z) \land OO(w,g)) \land$
$(C(y) \land \neg\, P(y,z) \longrightarrow Q(w,y,z) \land OO(w,b))))$
$\land$
$(\forall\, w.\ C(w) \land$
$(\forall\, y\ z.$
$(C(y) \land P(y,z) \longrightarrow Q(w,y,z) \land OO(w,g)) \land$
$(C(y) \land \neg\, P(y,z) \longrightarrow Q(w,y,z) \land OO(w,b))) \longrightarrow$
$(\exists\, v.\ C(v) \land$
$(\forall\, y.\ ((C(y) \land Q(w,y,y)) \land OO(w,g) \longrightarrow \neg\, P(v,y)) \land$
$((C(y) \land Q(w,y,y)) \land OO(w,b) \longrightarrow P(v,y) \land OO(v,b)))))$
$\longrightarrow \neg\, (\exists\, x.\ A(x) \land (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z))))$›
⟨*proof*⟩

Halting problem II: credited to M. Bruschi by Li Dafa in JAR 18(1), p. 105.

**lemma**
‹$((\exists\, x.\ A(x) \land (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z)))) \longrightarrow$
$(\exists\, w.\ C(w) \land (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(w,y,z)))))$
$\land$
$(\forall\, w.\ C(w) \land (\forall\, u.\ C(u) \longrightarrow (\forall\, v.\ D(w,u,v))) \longrightarrow$
$(\forall\, y\ z.$
$(C(y) \land P(y,z) \longrightarrow Q(w,y,z) \land OO(w,g)) \land$
$(C(y) \land \neg\, P(y,z) \longrightarrow Q(w,y,z) \land OO(w,b))))$
$\land$
$((\exists\, w.\ C(w) \land (\forall\, y.\ (C(y) \land P(y,y) \longrightarrow Q(w,y,y) \land OO(w,g)) \land$
$(C(y) \land \neg\, P(y,y) \longrightarrow Q(w,y,y) \land OO(w,b))))$
$\longrightarrow$
$(\exists\, v.\ C(v) \land (\forall\, y.\ (C(y) \land P(y,y) \longrightarrow P(v,y) \land OO(v,g)) \land$
$(C(y) \land \neg\, P(y,y) \longrightarrow P(v,y) \land OO(v,b)))))$
$\longrightarrow$
$((\exists\, v.\ C(v) \land (\forall\, y.\ (C(y) \land P(y,y) \longrightarrow P(v,y) \land OO(v,g)) \land$
$(C(y) \land \neg\, P(y,y) \longrightarrow P(v,y) \land OO(v,b))))$
$\longrightarrow$
$(\exists\, u.\ C(u) \land (\forall\, y.\ (C(y) \land P(y,y) \longrightarrow \neg\, P(u,y)) \land$
$(C(y) \land \neg\, P(y,y) \longrightarrow P(u,y) \land OO(u,b)))))$
$\longrightarrow \neg\, (\exists\, x.\ A(x) \land (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z))))$›
⟨*proof*⟩

Challenge found on info-hol.

**lemma** ‹$\forall\, x.\ \exists\, v\ w.\ \forall\, y\ z.\ P(x) \land Q(y) \longrightarrow (P(v) \lor R(w)) \land (R(z) \longrightarrow Q(v))$›
⟨*proof*⟩

Attributed to Lewis Carroll by S. G. Pulman. The first or last assumption can be deleted.

**lemma**
‹$(\forall\, x.\ \mathit{honest}(x) \land \mathit{industrious}(x) \longrightarrow \mathit{healthy}(x)) \land$

$\neg$ ($\exists$ *x*. *grocer*(*x*) $\wedge$ *healthy*(*x*)) $\wedge$
($\forall$ *x*. *industrious*(*x*) $\wedge$ *grocer*(*x*) $\longrightarrow$ *honest*(*x*)) $\wedge$
($\forall$ *x*. *cyclist*(*x*) $\longrightarrow$ *industrious*(*x*)) $\wedge$
($\forall$ *x*. $\neg$ *healthy*(*x*) $\wedge$ *cyclist*(*x*) $\longrightarrow$ $\neg$ *honest*(*x*))
$\longrightarrow$ ($\forall$ *x*. *grocer*(*x*) $\longrightarrow$ $\neg$ *cyclist*(*x*))›
⟨*proof*⟩

**end**

# 12 First-Order Logic: propositional examples (classical version)

**theory** *Propositional-Cla*
**imports** *FOL*
**begin**

commutative laws of $\wedge$ and $\vee$

**lemma** ‹$P \wedge Q \longrightarrow Q \wedge P$›
⟨*proof*⟩

**lemma** ‹$P \vee Q \longrightarrow Q \vee P$›
⟨*proof*⟩

associative laws of $\wedge$ and $\vee$

**lemma** ‹$(P \wedge Q) \wedge R \longrightarrow P \wedge (Q \wedge R)$›
⟨*proof*⟩

**lemma** ‹$(P \vee Q) \vee R \longrightarrow P \vee (Q \vee R)$›
⟨*proof*⟩

distributive laws of $\wedge$ and $\vee$

**lemma** ‹$(P \wedge Q) \vee R \longrightarrow (P \vee R) \wedge (Q \vee R)$›
⟨*proof*⟩

**lemma** ‹$(P \vee R) \wedge (Q \vee R) \longrightarrow (P \wedge Q) \vee R$›
⟨*proof*⟩

**lemma** ‹$(P \vee Q) \wedge R \longrightarrow (P \wedge R) \vee (Q \wedge R)$›
⟨*proof*⟩

**lemma** ‹$(P \wedge R) \vee (Q \wedge R) \longrightarrow (P \vee Q) \wedge R$›
⟨*proof*⟩

Laws involving implication

**lemma** ‹$(P \longrightarrow R) \wedge (Q \longrightarrow R) \longleftrightarrow (P \vee Q \longrightarrow R)$›

⟨*proof*⟩

**lemma** ‹$(P \wedge Q \longrightarrow R) \longleftrightarrow (P \longrightarrow (Q \longrightarrow R))$›
⟨*proof*⟩

**lemma** ‹$((P \longrightarrow R) \longrightarrow R) \longrightarrow ((Q \longrightarrow R) \longrightarrow R) \longrightarrow (P \wedge Q \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

**lemma** ‹$\neg (P \longrightarrow R) \longrightarrow \neg (Q \longrightarrow R) \longrightarrow \neg (P \wedge Q \longrightarrow R)$›
⟨*proof*⟩

**lemma** ‹$(P \longrightarrow Q \wedge R) \longleftrightarrow (P \longrightarrow Q) \wedge (P \longrightarrow R)$›
⟨*proof*⟩

Propositions-as-types

**lemma** ‹$P \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩
**lemma** ‹$(P \longrightarrow Q \longrightarrow R) \longrightarrow (P \longrightarrow Q) \longrightarrow (P \longrightarrow R)$›
⟨*proof*⟩
**lemma** ‹$(P \longrightarrow Q) \vee (P \longrightarrow R) \longrightarrow (P \longrightarrow Q \vee R)$›
⟨*proof*⟩

**lemma** ‹$(P \longrightarrow Q) \longrightarrow (\neg Q \longrightarrow \neg P)$›
⟨*proof*⟩

Schwichtenberg's examples (via T. Nipkow)

**lemma** *stab-imp*: ‹$(((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow R) \longrightarrow P \longrightarrow Q$›
⟨*proof*⟩

**lemma** *stab-to-peirce*:
‹$(((P \longrightarrow R) \longrightarrow R) \longrightarrow P) \longrightarrow (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q)$
$\longrightarrow ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

**lemma** *peirce-imp1*:
‹$(((Q \longrightarrow R) \longrightarrow Q) \longrightarrow Q)$
$\longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow P \longrightarrow Q) \longrightarrow P \longrightarrow Q$›
⟨*proof*⟩

**lemma** *peirce-imp2*: ‹$(((P \longrightarrow R) \longrightarrow P) \longrightarrow P) \longrightarrow ((P \longrightarrow Q \longrightarrow R) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

**lemma** *mints*: ‹$((((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \longrightarrow Q) \longrightarrow Q$›
⟨*proof*⟩

**lemma** *mints-solovev*: ‹$(P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

**lemma** *tatsuta*:
‹(((*P7* ⟶ *P1*) ⟶ *P10*) ⟶ *P4* ⟶ *P5*)
⟶ (((*P8* ⟶ *P2*) ⟶ *P9*) ⟶ *P3* ⟶ *P10*)
⟶ (*P1* ⟶ *P8*) ⟶ *P6* ⟶ *P7*
⟶ (((*P3* ⟶ *P2*) ⟶ *P9*) ⟶ *P4*)
⟶ (*P1* ⟶ *P3*) ⟶ (((*P6* ⟶ *P1*) ⟶ *P2*) ⟶ *P9*) ⟶ *P5*›
⟨*proof*⟩

**lemma** *tatsuta1*:
‹(((*P8* ⟶ *P2*) ⟶ *P9*) ⟶ *P3* ⟶ *P10*)
⟶ (((*P3* ⟶ *P2*) ⟶ *P9*) ⟶ *P4*)
⟶ (((*P6* ⟶ *P1*) ⟶ *P2*) ⟶ *P9*)
⟶ (((*P7* ⟶ *P1*) ⟶ *P10*) ⟶ *P4* ⟶ *P5*)
⟶ (*P1* ⟶ *P3*) ⟶ (*P1* ⟶ *P8*) ⟶ *P6* ⟶ *P7* ⟶ *P5*›
⟨*proof*⟩

**end**

# 13 First-Order Logic: quantifier examples (classical version)

**theory** *Quantifiers-Cla*
**imports** *FOL*
**begin**

**lemma** ‹(∀ *x y*. *P*(*x*,*y*)) ⟶ (∀ *y x*. *P*(*x*,*y*))›
⟨*proof*⟩

**lemma** ‹(∃ *x y*. *P*(*x*,*y*)) ⟶ (∃ *y x*. *P*(*x*,*y*))›
⟨*proof*⟩

Converse is false.

**lemma** ‹(∀ *x*. *P*(*x*)) ∨ (∀ *x*. *Q*(*x*)) ⟶ (∀ *x*. *P*(*x*) ∨ *Q*(*x*))›
⟨*proof*⟩

**lemma** ‹(∀ *x*. *P* ⟶ *Q*(*x*)) ⟷ (*P* ⟶ (∀ *x*. *Q*(*x*)))›
⟨*proof*⟩

**lemma** ‹(∀ *x*. *P*(*x*) ⟶ *Q*) ⟷ ((∃ *x*. *P*(*x*)) ⟶ *Q*)›
⟨*proof*⟩

Some harder ones.

**lemma** ‹(∃ *x*. *P*(*x*) ∨ *Q*(*x*)) ⟷ (∃ *x*. *P*(*x*)) ∨ (∃ *x*. *Q*(*x*))›
⟨*proof*⟩
**lemma** ‹(∃ *x*. *P*(*x*) ∧ *Q*(*x*)) ⟶ (∃ *x*. *P*(*x*)) ∧ (∃ *x*. *Q*(*x*))›
⟨*proof*⟩

Basic test of quantifier reasoning.

**lemma** ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

The following should fail, as they are false!

**lemma** ‹$(\forall x.\ \exists y.\ Q(x,y)) \longrightarrow (\exists y.\ \forall x.\ Q(x,y))$›
⟨*proof*⟩

**lemma** ‹$(\exists x.\ Q(x)) \longrightarrow (\forall x.\ Q(x))$›
⟨*proof*⟩

**schematic-goal** ‹$P(?a) \longrightarrow (\forall x.\ P(x))$›
⟨*proof*⟩

**schematic-goal** ‹$(P(?a) \longrightarrow (\forall x.\ Q(x))) \longrightarrow (\forall x.\ P(x) \longrightarrow Q(x))$›
⟨*proof*⟩

Back to things that are provable . . .

**lemma** ‹$(\forall x.\ P(x) \longrightarrow Q(x)) \wedge (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

An example of why *exI* should be delayed as long as possible.

**lemma** ‹$(P \longrightarrow (\exists x.\ Q(x))) \wedge P \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

**schematic-goal** ‹$(\forall x.\ P(x) \longrightarrow Q(f(x))) \wedge (\forall x.\ Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a)$›
⟨*proof*⟩

**lemma** ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

Some slow ones

Principia Mathematica *11.53

**lemma** ‹$(\forall x\ y.\ P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow (\forall y.\ Q(y)))$›
⟨*proof*⟩

**lemma** ‹$(\exists x\ y.\ P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x.\ P(x) \wedge (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

**lemma** ‹$(\exists y.\ \forall x.\ P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x.\ P(x) \longrightarrow (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

**end**

**theory** *Miniscope*
**imports** *FOL*
**begin**

**lemmas** *ccontr* = *FalseE* [*THEN classical*]

## 13.1 Negation Normal Form

### 13.1.1 de Morgan laws

**lemma** *demorgans1*:
‹$\neg (P \wedge Q) \longleftrightarrow \neg P \vee \neg Q$›
‹$\neg (P \vee Q) \longleftrightarrow \neg P \wedge \neg Q$›
‹$\neg \neg P \longleftrightarrow P$›
⟨*proof*⟩

**lemma** *demorgans2*:
‹$\bigwedge P.\ \neg (\forall x.\ P(x)) \longleftrightarrow (\exists x.\ \neg P(x))$›
‹$\bigwedge P.\ \neg (\exists x.\ P(x)) \longleftrightarrow (\forall x.\ \neg P(x))$›
⟨*proof*⟩

**lemmas** *demorgans* = *demorgans1 demorgans2*

**lemma** *nnf-simps*:
‹$(P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q)$›
‹$\neg (P \longrightarrow Q) \longleftrightarrow (P \wedge \neg Q)$›
‹$(P \longleftrightarrow Q) \longleftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$›
‹$\neg (P \longleftrightarrow Q) \longleftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$›
⟨*proof*⟩

### 13.1.2 Pushing in the existential quantifiers

**lemma** *ex-simps*:
‹$(\exists x.\ P) \longleftrightarrow P$›
‹$\bigwedge P\ Q.\ (\exists x.\ P(x) \wedge Q) \longleftrightarrow (\exists x.\ P(x)) \wedge Q$›
‹$\bigwedge P\ Q.\ (\exists x.\ P \wedge Q(x)) \longleftrightarrow P \wedge (\exists x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\exists x.\ P(x) \vee Q(x)) \longleftrightarrow (\exists x.\ P(x)) \vee (\exists x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\exists x.\ P(x) \vee Q) \longleftrightarrow (\exists x.\ P(x)) \vee Q$›
‹$\bigwedge P\ Q.\ (\exists x.\ P \vee Q(x)) \longleftrightarrow P \vee (\exists x.\ Q(x))$›
⟨*proof*⟩

### 13.1.3 Pushing in the universal quantifiers

**lemma** *all-simps*:

‹$(\forall x.\ P) \longleftrightarrow P$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \wedge Q(x)) \longleftrightarrow (\forall x.\ P(x)) \wedge (\forall x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \wedge Q) \longleftrightarrow (\forall x.\ P(x)) \wedge Q$›
‹$\bigwedge P\ Q.\ (\forall x.\ P \wedge Q(x)) \longleftrightarrow P \wedge (\forall x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \vee Q) \longleftrightarrow (\forall x.\ P(x)) \vee Q$›
‹$\bigwedge P\ Q.\ (\forall x.\ P \vee Q(x)) \longleftrightarrow P \vee (\forall x.\ Q(x))$›
⟨*proof*⟩

**lemmas** *mini-simps* = *demorgans nnf-simps ex-simps all-simps*

⟨*ML*⟩

**end**

# 14 First-Order Logic: the 'if' example

**theory** *If*
**imports** *FOL*
**begin**

**definition** *if* :: ‹[*o*,*o*,*o*]=>*o*›
  **where** ‹$if(P,Q,R) \equiv P \wedge Q \vee \neg P \wedge R$›

**lemma** *ifI*: ‹$[\![P \Longrightarrow Q;\ \neg P \Longrightarrow R]\!] \Longrightarrow if(P,Q,R)$›
⟨*proof*⟩

**lemma** *ifE*: ‹$[\![if(P,Q,R);\ [\![P;\ Q]\!] \Longrightarrow S;\ [\![\neg P;\ R]\!] \Longrightarrow S]\!] \Longrightarrow S$›
⟨*proof*⟩

**lemma** *if-commute*: ‹$if(P,\ if(Q,A,B),\ if(Q,C,D)) \longleftrightarrow if(Q,\ if(P,A,C),\ if(P,B,D))$›
⟨*proof*⟩

Trying again from the beginning in order to use *blast*

**declare** *ifI* [*intro*!]
**declare** *ifE* [*elim*!]

**lemma** *if-commute*: ‹$if(P,\ if(Q,A,B),\ if(Q,C,D)) \longleftrightarrow if(Q,\ if(P,A,C),\ if(P,B,D))$›
⟨*proof*⟩

**lemma** ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,A,B))$›
⟨*proof*⟩

Trying again from the beginning in order to prove from the definitions

**lemma** ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,A,B))$›
⟨*proof*⟩

An invalid formula. High-level rules permit a simpler diagnosis.

**lemma** ‹$if(if(P,Q,R), A, B) \longleftrightarrow if(P, if(Q,A,B), if(R,B,A))$›
⟨*proof*⟩

Trying again from the beginning in order to prove from the definitions.

**lemma** ‹$if(if(P,Q,R), A, B) \longleftrightarrow if(P, if(Q,A,B), if(R,B,A))$›
⟨*proof*⟩

**end**