ZF

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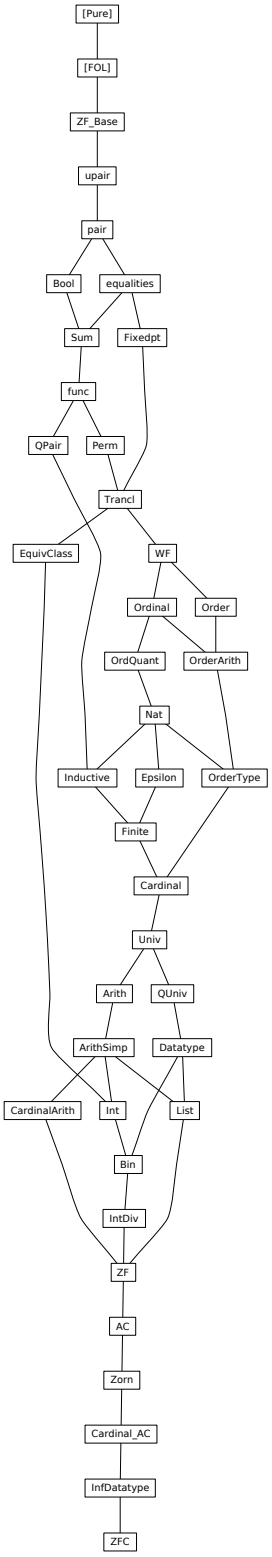
Contents


[Pure]
[FOL]
ZF_Base
upair
pair
Bool
equalities
Sum
Fixedpt
func
QPair
Perm
Trancl
EquivClass
WF
Ordinal
Order
OrdQuant
OrderArith
Nat
Inductive
Epsilon
OrderType
Finite
Cardinal
Univ
Arith
QUniv
ArithSimp
Datatype
CardinalArith
Int
List
Bin
IntDiv
ZF
AC
Zorn
Cardinal_AC
InfDatatype
ZFC

1 Base of Zermelo-Fraenkel Set Theory

theory *ZF-Base*
imports *FOL*
begin

1.1 Signature

declare [[*eta-contract* = *false*]]

typedecl *i*
instance *i* :: *term* ⟨*proof*⟩

axiomatization *mem* :: [*i*, *i*] ⇒ *o* (**infixl** ‹∈› *50*) — membership relation
and *zero* :: *i* (‹*0*›) — the empty set
and *Pow* :: *i* ⇒ *i* — power sets
and *Inf* :: *i* — infinite set
and *Union* :: *i* ⇒ *i* (‹⋃ -› [*90*] *90*)
and *PrimReplace* :: [*i*, [*i*, *i*] ⇒ *o*] ⇒ *i*

abbreviation *not-mem* :: [*i*, *i*] ⇒ *o* (**infixl** ‹∉› *50*) — negated membership relation
where *x* ∉ *y* ≡ ¬ (*x* ∈ *y*)

1.2 Bounded Quantifiers

definition *Ball* :: [*i*, *i* ⇒ *o*] ⇒ *o*
where *Ball*(*A*, *P*) ≡ ∀ *x*. *x*∈*A* ⟶ *P*(*x*)

definition *Bex* :: [*i*, *i* ⇒ *o*] ⇒ *o*
where *Bex*(*A*, *P*) ≡ ∃ *x*. *x*∈*A* ∧ *P*(*x*)

syntax
-Ball :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(*3*∀ -∈-./ -)› *10*)
-Bex :: [*pttrn*, *i*, *o*] ⇒ *o* (‹(*3*∃ -∈-./ -)› *10*)
translations
∀ *x*∈*A*. *P* ⇌ *CONST Ball*(*A*, λ*x*. *P*)
∃ *x*∈*A*. *P* ⇌ *CONST Bex*(*A*, λ*x*. *P*)

1.3 Variations on Replacement

definition *Replace* :: [*i*, [*i*, *i*] ⇒ *o*] ⇒ *i*
where *Replace*(*A*,*P*) ≡ *PrimReplace*(*A*, λ*x y*. (∃!*z*. *P*(*x*,*z*)) ∧ *P*(*x*,*y*))

syntax
-Replace :: [*pttrn*, *pttrn*, *i*, *o*] ⇒ *i* (‹(*1*{- ./ - ∈ -, -})›)
translations
{*y*. *x*∈*A*, *Q*} ⇌ *CONST Replace*(*A*, λ*x y*. *Q*)

definition *RepFun* :: $[i,\ i \Rightarrow i] \Rightarrow i$
  where $RepFun(A,f) \equiv \{y\ .\ x{\in}A,\ y{=}f(x)\}$

syntax
  -RepFun :: $[i,\ pttrn,\ i] \Rightarrow i$ $(\langle(1\{\text{-}\ ./\ \text{-}\in\text{-}\})\rangle\ [51,0,51])$
translations
  $\{b.\ x{\in}A\} \rightleftharpoons CONST\ RepFun(A,\ \lambda x.\ b)$

definition *Collect* :: $[i,\ i \Rightarrow o] \Rightarrow i$
  where $Collect(A,P) \equiv \{y\ .\ x{\in}A,\ x{=}y \wedge P(x)\}$

syntax
  -Collect :: $[pttrn,\ i,\ o] \Rightarrow i$ $(\langle(1\{\text{-}\in\text{-}\ ./\ \text{-}\})\rangle)$
translations
  $\{x{\in}A.\ P\} \rightleftharpoons CONST\ Collect(A,\ \lambda x.\ P)$

1.4 General union and intersection

definition *Inter* :: $i \Rightarrow i$ $(\langle\bigcap\text{-}\rangle\ [90]\ 90)$
  where $\bigcap(A) \equiv \{\ x{\in}\bigcup(A)\ .\ \forall y{\in}A.\ x{\in}y\}$

syntax
  -UNION :: $[pttrn,\ i,\ i] \Rightarrow i$ $(\langle(3\bigcup\text{-}{\in}\text{-}./\ \text{-})\rangle\ 10)$
  -INTER :: $[pttrn,\ i,\ i] \Rightarrow i$ $(\langle(3\bigcap\text{-}{\in}\text{-}./\ \text{-})\rangle\ 10)$
translations
  $\bigcup x{\in}A.\ B == CONST\ Union(\{B.\ x{\in}A\})$
  $\bigcap x{\in}A.\ B == CONST\ Inter(\{B.\ x{\in}A\})$

1.5 Finite sets and binary operations

definition *Upair* :: $[i,\ i] \Rightarrow i$
  where $Upair(a,b) \equiv \{y.\ x{\in}Pow(Pow(0)),\ (x{=}0 \wedge y{=}a) \mid (x{=}Pow(0) \wedge y{=}b)\}$

definition *Subset* :: $[i,\ i] \Rightarrow o$ (**infixl** $\langle\subseteq\rangle$ *50*) — subset relation
  where *subset-def*: $A \subseteq B \equiv \forall x{\in}A.\ x{\in}B$

definition *Diff* :: $[i,\ i] \Rightarrow i$ (**infixl** $\langle-\rangle$ *65*) — set difference
  where $A - B \equiv \{\ x{\in}A\ .\ \neg(x{\in}B)\ \}$

definition *Un* :: $[i,\ i] \Rightarrow i$ (**infixl** $\langle\cup\rangle$ *65*) — binary union
  where $A \cup B \equiv \bigcup(Upair(A,B))$

definition *Int* :: $[i,\ i] \Rightarrow i$ (**infixl** $\langle\cap\rangle$ *70*) — binary intersection
  where $A \cap B \equiv \bigcap(Upair(A,B))$

definition *cons* :: $[i,\ i] \Rightarrow i$

where $cons(a,A) \equiv Upair(a,a) \cup A$

definition $succ :: i \Rightarrow i$
where $succ(i) \equiv cons(i,\ i)$

nonterminal *is*
syntax
$:: i \Rightarrow is$ $(\langle\text{-}\rangle)$
$\text{-}Enum :: [i,\ is] \Rightarrow is$ $(\langle\text{-},/\ \text{-}\rangle)$
$\text{-}Finset :: is \Rightarrow i$ $(\langle\{(\text{-})\}\rangle)$
translations
$\{x,\ xs\} == CONST\ cons(x,\ \{xs\})$
$\{x\} == CONST\ cons(x,\ 0)$

1.6 Axioms

axiomatization
where
extension: $A = B \longleftrightarrow A \subseteq B \wedge B \subseteq A$ **and**
Union-iff: $A \in \bigcup(C) \longleftrightarrow (\exists\, B{\in}C.\ A{\in}B)$ **and**
Pow-iff: $A \in Pow(B) \longleftrightarrow A \subseteq B$ **and**

infinity: $0 \in Inf \wedge (\forall\, y{\in}Inf.\ succ(y) \in Inf)$ **and**

foundation: $A = 0 \vee (\exists\, x{\in}A.\ \forall\, y{\in}x.\ y{\notin}A)$ **and**

replacement: $(\forall\, x{\in}A.\ \forall\, y\ z.\ P(x,y) \wedge P(x,z) \longrightarrow y = z) \Longrightarrow$
$b \in PrimReplace(A,P) \longleftrightarrow (\exists\, x{\in}A.\ P(x,b))$

1.7 Definite descriptions – via Replace over the set "1"

definition $The :: (i \Rightarrow o) \Rightarrow i$ (**binder** $\langle THE\ \rangle\ 10$)
where *the-def*: $The(P) \equiv \bigcup(\{y\ .\ x \in \{0\},\ P(y)\})$

definition $If :: [o,\ i,\ i] \Rightarrow i$ $(\langle(if\ (\text{-})/\ then\ (\text{-})/\ else\ (\text{-}))\rangle\ [10]\ 10)$
where *if-def*: *if P then a else b* $\equiv THE\ z.\ P \wedge z{=}a \mid \neg P \wedge z{=}b$

abbreviation (*input*)
$old\text{-}if :: [o,\ i,\ i] \Rightarrow i$ $(\langle if\ '(\text{-},\text{-},\text{-}')\rangle)$
where $if(P,a,b) \equiv If(P,a,b)$

1.8 Ordered Pairing

definition $Pair :: [i,\ i] \Rightarrow i$
where $Pair(a,b) \equiv \{\{a,a\},\ \{a,b\}\}$

definition $fst :: i \Rightarrow i$

where $fst(p) \equiv THE\ a.\ \exists\, b.\ p = Pair(a,\ b)$

definition $snd :: i \Rightarrow i$
where $snd(p) \equiv THE\ b.\ \exists\, a.\ p = Pair(a,\ b)$

definition $split :: [[i,\ i] \Rightarrow 'a,\ i] \Rightarrow 'a::\{\}$ — for pattern-matching
where $split(c) \equiv \lambda p.\ c(fst(p),\ snd(p))$

nonterminal *patterns*
syntax
-pattern :: *patterns* ⇒ *pttrn* (‹⟨-⟩›)
:: *pttrn* ⇒ *patterns* (‹-›)
-patterns :: [*pttrn*, *patterns*] ⇒ *patterns* (‹-,/-›)
-Tuple :: [*i*, *is*] ⇒ *i* (‹⟨(-,/ -)⟩›)
translations
$\langle x,\ y,\ z\rangle == \langle x,\ \langle y,\ z\rangle\rangle$
$\langle x,\ y\rangle == CONST\ Pair(x,\ y)$
$\lambda\langle x,y,zs\rangle.b == CONST\ split(\lambda x\ \langle y,zs\rangle.b)$
$\lambda\langle x,y\rangle.b == CONST\ split(\lambda x\ y.\ b)$

definition $Sigma :: [i,\ i \Rightarrow i] \Rightarrow i$
where $Sigma(A,B) \equiv \bigcup x{\in}A.\ \bigcup y{\in}B(x).\ \{\langle x,y\rangle\}$

abbreviation *cart-prod* :: [*i*, *i*] ⇒ *i* (**infixr** ‹×› *80*) — Cartesian product
where $A \times B \equiv Sigma(A,\ \lambda\text{-}.\ B)$

1.9 Relations and Functions

definition $converse :: i \Rightarrow i$
where $converse(r) \equiv \{z.\ w{\in}r,\ \exists\, x\ y.\ w{=}\langle x,y\rangle \wedge z{=}\langle y,x\rangle\}$

definition $domain :: i \Rightarrow i$
where $domain(r) \equiv \{x.\ w{\in}r,\ \exists\, y.\ w{=}\langle x,y\rangle\}$

definition $range :: i \Rightarrow i$
where $range(r) \equiv domain(converse(r))$

definition $field :: i \Rightarrow i$
where $field(r) \equiv domain(r) \cup range(r)$

definition $relation :: i \Rightarrow o$ — recognizes sets of pairs
where $relation(r) \equiv \forall\, z{\in}r.\ \exists\, x\ y.\ z = \langle x,y\rangle$

definition $function :: i \Rightarrow o$ — recognizes functions; can have non-pairs
where $function(r) \equiv \forall\, x\ y.\ \langle x,y\rangle \in r \longrightarrow (\forall\, y'.\ \langle x,y'\rangle \in r \longrightarrow y = y')$

definition $Image :: [i,\ i] \Rightarrow i$ (**infixl** ‹‘› *90*) — image
where *image-def*: $r\ `` A \equiv \{y \in range(r).\ \exists\, x{\in}A.\ \langle x,y\rangle \in r\}$

definition *vimage* :: $[i,\ i] \Rightarrow i$ (**infixl** ‹$-``$› *90*) — inverse image
where *vimage-def*: $r -`` A \equiv converse(r)``A$

definition *restrict* :: $[i,\ i] \Rightarrow i$
where $restrict(r,A) \equiv \{z \in r.\ \exists x{\in}A.\ \exists y.\ z = \langle x,y\rangle\}$

definition *Lambda* :: $[i,\ i \Rightarrow i] \Rightarrow i$
where *lam-def*: $Lambda(A,b) \equiv \{\langle x,b(x)\rangle.\ x{\in}A\}$

definition *apply* :: $[i,\ i] \Rightarrow i$ (**infixl** ‹`› *90*) — function application
where $f`a \equiv \bigcup(f``\{a\})$

definition *Pi* :: $[i,\ i \Rightarrow i] \Rightarrow i$
where $Pi(A,B) \equiv \{f{\in}Pow(Sigma(A,B)).\ A{\subseteq}domain(f) \wedge function(f)\}$

abbreviation *function-space* :: $[i,\ i] \Rightarrow i$ (**infixr** ‹$\rightarrow$› *60*) — function space
where $A \rightarrow B \equiv Pi(A,\ \lambda\text{-}.\ B)$

syntax
-PROD :: $[pttrn,\ i,\ i] \Rightarrow i$ (‹$(3\prod$ -$\in$-./ -)› *10*)
-SUM :: $[pttrn,\ i,\ i] \Rightarrow i$ (‹$(3\sum$ -$\in$-./ -)› *10*)
-lam :: $[pttrn,\ i,\ i] \Rightarrow i$ (‹$(3\lambda$-$\in$-./ -)› *10*)
translations
$\prod x{\in}A.\ B$ == *CONST Pi*$(A,\ \lambda x.\ B)$
$\sum x{\in}A.\ B$ == *CONST Sigma*$(A,\ \lambda x.\ B)$
$\lambda x{\in}A.\ f$ == *CONST Lambda*$(A,\ \lambda x.\ f)$

1.10 ASCII syntax

notation (*ASCII*)
cart-prod (**infixr** ‹*› *80*) **and**
Int (**infixl** ‹*Int› *70*) **and**
Un (**infixl** ‹*Un› *65*) **and**
function-space (**infixr** ‹->› *60*) **and**
Subset (**infixl** ‹<=› *50*) **and**
mem (**infixl** ‹:› *50*) **and**
not-mem (**infixl** ‹~:› *50*)

syntax (*ASCII*)
-Ball :: $[pttrn,\ i,\ o] \Rightarrow o$ (‹$(3$*ALL* -:-./ -)› *10*)
-Bex :: $[pttrn,\ i,\ o] \Rightarrow o$ (‹$(3$*EX* -:-./ -)› *10*)

-Collect :: [*pttrn*, *i*, *o*] ⇒ *i* (‹(*1*{-: - ./ -})›)
-Replace :: [*pttrn*, *pttrn*, *i*, *o*] ⇒ *i* (‹(*1*{- ./ -: -, -})›)
-RepFun :: [*i*, *pttrn*, *i*] ⇒ *i* (‹(*1*{- ./ -: -})› [*51*,*0*,*51*])
-UNION :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(*3UN* -:-./ -)› *10*)
-INTER :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(*3INT* -:-./ -)› *10*)
-PROD :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(*3PROD* -:-./ -)› *10*)
-SUM :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(*3SUM* -:-./ -)› *10*)
-lam :: [*pttrn*, *i*, *i*] ⇒ *i* (‹(*3lam* -:-./ -)› *10*)
-Tuple :: [*i*, *is*] ⇒ *i* (‹<(-,/ -)>›)
-pattern :: *patterns* ⇒ *pttrn* (‹<->›)

1.11 Substitution

lemma *subst-elem*: ⟦*b*∈*A*; *a*=*b*⟧ ⟹ *a*∈*A*
⟨*proof*⟩

1.12 Bounded universal quantifier

lemma *ballI* [*intro*!]: ⟦⋀*x*. *x*∈*A* ⟹ *P*(*x*)⟧ ⟹ ∀ *x*∈*A*. *P*(*x*)
⟨*proof*⟩

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: ⟦∀ *x*∈*A*. *P*(*x*); *x*: *A*⟧ ⟹ *P*(*x*)
⟨*proof*⟩

lemma *rev-ballE* [*elim*]:
⟦∀ *x*∈*A*. *P*(*x*); *x*∉*A* ⟹ *Q*; *P*(*x*) ⟹ *Q*⟧ ⟹ *Q*
⟨*proof*⟩

lemma *ballE*: ⟦∀ *x*∈*A*. *P*(*x*); *P*(*x*) ⟹ *Q*; *x*∉*A* ⟹ *Q*⟧ ⟹ *Q*
⟨*proof*⟩

lemma *rev-bspec*: ⟦*x*: *A*; ∀ *x*∈*A*. *P*(*x*)⟧ ⟹ *P*(*x*)
⟨*proof*⟩

lemma *ball-triv* [*simp*]: (∀ *x*∈*A*. *P*) ⟷ ((∃ *x*. *x*∈*A*) ⟶ *P*)
⟨*proof*⟩

lemma *ball-cong* [*cong*]:
⟦*A*=*A*′; ⋀*x*. *x*∈*A*′ ⟹ *P*(*x*) ⟷ *P*′(*x*)⟧ ⟹ (∀ *x*∈*A*. *P*(*x*)) ⟷ (∀ *x*∈*A*′. *P*′(*x*))
⟨*proof*⟩

lemma *atomize-ball*:
(⋀*x*. *x* ∈ *A* ⟹ *P*(*x*)) ≡ *Trueprop* (∀ *x*∈*A*. *P*(*x*))
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

1.13 Bounded existential quantifier

lemma *bexI* [*intro*]: $[\![P(x);\ x\colon A]\!] \Longrightarrow \exists\, x{\in}A.\ P(x)$
⟨*proof*⟩

lemma *rev-bexI*: $[\![x{\in}A;\ P(x)]\!] \Longrightarrow \exists\, x{\in}A.\ P(x)$
⟨*proof*⟩

lemma *bexCI*: $[\![\forall\, x{\in}A.\ \neg P(x) \Longrightarrow P(a);\ a\colon A]\!] \Longrightarrow \exists\, x{\in}A.\ P(x)$
⟨*proof*⟩

lemma *bexE* [*elim*!]: $[\![\exists\, x{\in}A.\ P(x);\ \bigwedge x.\ [\![x{\in}A;\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *bex-triv* [*simp*]: $(\exists\, x{\in}A.\ P) \longleftrightarrow ((\exists\, x.\ x{\in}A) \wedge P)$
⟨*proof*⟩

lemma *bex-cong* [*cong*]:
$[\![A{=}A';\ \bigwedge x.\ x{\in}A' \Longrightarrow P(x) \longleftrightarrow P'(x)]\!]$
$\Longrightarrow (\exists\, x{\in}A.\ P(x)) \longleftrightarrow (\exists\, x{\in}A'.\ P'(x))$
⟨*proof*⟩

1.14 Rules for subsets

lemma *subsetI* [*intro*!]:
$(\bigwedge x.\ x{\in}A \Longrightarrow x{\in}B) \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *subsetD* [*elim*]: $[\![A \subseteq B;\ c{\in}A]\!] \Longrightarrow c{\in}B$
⟨*proof*⟩

lemma *subsetCE* [*elim*]:
$[\![A \subseteq B;\ c{\notin}A \Longrightarrow P;\ c{\in}B \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *rev-subsetD*: $[\![c{\in}A;\ A{\subseteq}B]\!] \Longrightarrow c{\in}B$
⟨*proof*⟩

lemma *contra-subsetD*: $[\![A \subseteq B;\ c \notin B]\!] \Longrightarrow c \notin A$
⟨*proof*⟩

lemma *rev-contra-subsetD*: $[\![c \notin B;\ \ A \subseteq B]\!] \Longrightarrow c \notin A$
⟨*proof*⟩

lemma *subset-refl* [*simp*]: $A \subseteq A$
⟨*proof*⟩

lemma *subset-trans*: $[\![A{\subseteq}B;\ \ B{\subseteq}C]\!] \Longrightarrow A{\subseteq}C$
⟨*proof*⟩

lemma *subset-iff*:
$A{\subseteq}B \longleftrightarrow (\forall x.\ x{\in}A \longrightarrow x{\in}B)$
⟨*proof*⟩

For calculations

declare *subsetD* [*trans*] *rev-subsetD* [*trans*] *subset-trans* [*trans*]

1.15 Rules for equality

lemma *equalityI* [*intro*]: $[\![A \subseteq B;\ \ B \subseteq A]\!] \Longrightarrow A = B$
⟨*proof*⟩

lemma *equality-iffI*: $(\bigwedge x.\ x{\in}A \longleftrightarrow x{\in}B) \Longrightarrow A = B$
⟨*proof*⟩

lemmas *equalityD1* = *extension* [*THEN iffD1*, *THEN conjunct1*]
lemmas *equalityD2* = *extension* [*THEN iffD1*, *THEN conjunct2*]

lemma *equalityE*: $[\![A = B;\ \ [\![A{\subseteq}B;\ B{\subseteq}A]\!] \Longrightarrow P]\!] \Longrightarrow\ \ P$
⟨*proof*⟩

lemma *equalityCE*:
$[\![A = B;\ \ [\![c{\in}A;\ c{\in}B]\!] \Longrightarrow P;\ \ [\![c{\notin}A;\ c{\notin}B]\!] \Longrightarrow P]\!] \Longrightarrow\ \ P$
⟨*proof*⟩

lemma *equality-iffD*:
$A = B \Longrightarrow (\bigwedge x.\ x \in A \longleftrightarrow x \in B)$
⟨*proof*⟩

1.16 Rules for Replace – the derived form of replacement

lemma *Replace-iff*:
$b \in \{y.\ x{\in}A,\ P(x,y)\} \ \longleftrightarrow\ (\exists\, x{\in}A.\ P(x,b) \wedge (\forall\, y.\ P(x,y) \longrightarrow y{=}b))$
⟨*proof*⟩

lemma *ReplaceI* [*intro*]:

$[\![P(x,b);\ x\colon A;\ \bigwedge y.\ P(x,y) \Longrightarrow y{=}b]\!] \Longrightarrow$
$b \in \{y.\ x{\in}A,\ P(x,y)\}$
⟨*proof*⟩

lemma *ReplaceE*:
$[\![b \in \{y.\ x{\in}A,\ P(x,y)\};$
$\bigwedge x.\ [\![x\colon A;\ P(x,b);\ \forall y.\ P(x,y) \longrightarrow y{=}b]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *ReplaceE2* [*elim*!]:
$[\![b \in \{y.\ x{\in}A,\ P(x,y)\};$
$\bigwedge x.\ [\![x\colon A;\ P(x,b)]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *Replace-cong* [*cong*]:
$[\![A{=}B;\ \bigwedge x\ y.\ x{\in}B \Longrightarrow P(x,y) \longleftrightarrow Q(x,y)]\!] \Longrightarrow Replace(A,P) = Replace(B,Q)$
⟨*proof*⟩

1.17 Rules for RepFun

lemma *RepFunI*: $a \in A \Longrightarrow f(a) \in \{f(x).\ x{\in}A\}$
⟨*proof*⟩

lemma *RepFun-eqI* [*intro*]: $[\![b{=}f(a);\ a \in A]\!] \Longrightarrow b \in \{f(x).\ x{\in}A\}$
⟨*proof*⟩

lemma *RepFunE* [*elim*!]:
$[\![b \in \{f(x).\ x{\in}A\};$
$\bigwedge x.[\![x{\in}A;\ b{=}f(x)]\!] \Longrightarrow P]\!] \Longrightarrow$
P
⟨*proof*⟩

lemma *RepFun-cong* [*cong*]:
$[\![A{=}B;\ \bigwedge x.\ x{\in}B \Longrightarrow f(x){=}g(x)]\!] \Longrightarrow RepFun(A,f) = RepFun(B,g)$
⟨*proof*⟩

lemma *RepFun-iff* [*simp*]: $b \in \{f(x).\ x{\in}A\} \longleftrightarrow (\exists\, x{\in}A.\ b{=}f(x))$
⟨*proof*⟩

lemma *triv-RepFun* [*simp*]: $\{x.\ x{\in}A\} = A$
⟨*proof*⟩

1.18 Rules for Collect – forming a subset by separation

lemma *separation* [*simp*]: $a \in \{x{\in}A.\ P(x)\} \longleftrightarrow a{\in}A \wedge P(a)$

⟨*proof*⟩

lemma *CollectI* [*intro*!]: ⟦*a*∈*A*; *P*(*a*)⟧ ⟹ *a* ∈ {*x*∈*A*. *P*(*x*)}
⟨*proof*⟩

lemma *CollectE* [*elim*!]: ⟦*a* ∈ {*x*∈*A*. *P*(*x*)}; ⟦*a*∈*A*; *P*(*a*)⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

lemma *CollectD1*: *a* ∈ {*x*∈*A*. *P*(*x*)} ⟹ *a*∈*A* **and** *CollectD2*: *a* ∈ {*x*∈*A*. *P*(*x*)}
⟹ *P*(*a*)
⟨*proof*⟩

lemma *Collect-cong* [*cong*]:
⟦*A*=*B*; ⋀*x*. *x*∈*B* ⟹ *P*(*x*) ⟷ *Q*(*x*)⟧
⟹ *Collect*(*A*, λ*x*. *P*(*x*)) = *Collect*(*B*, λ*x*. *Q*(*x*))
⟨*proof*⟩

1.19 Rules for Unions

declare *Union-iff* [*simp*]

lemma *UnionI* [*intro*]: ⟦*B*: *C*; *A*: *B*⟧ ⟹ *A*: ⋃(*C*)
⟨*proof*⟩

lemma *UnionE* [*elim*!]: ⟦*A* ∈ ⋃(*C*); ⋀*B*.⟦*A*: *B*; *B*: *C*⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

1.20 Rules for Unions of families

lemma *UN-iff* [*simp*]: *b* ∈ (⋃*x*∈*A*. *B*(*x*)) ⟷ (∃*x*∈*A*. *b* ∈ *B*(*x*))
⟨*proof*⟩

lemma *UN-I*: ⟦*a*: *A*; *b*: *B*(*a*)⟧ ⟹ *b*: (⋃*x*∈*A*. *B*(*x*))
⟨*proof*⟩

lemma *UN-E* [*elim*!]:
⟦*b* ∈ (⋃*x*∈*A*. *B*(*x*)); ⋀*x*.⟦*x*: *A*; *b*: *B*(*x*)⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

lemma *UN-cong*:
⟦*A*=*B*; ⋀*x*. *x*∈*B* ⟹ *C*(*x*)=*D*(*x*)⟧ ⟹ (⋃*x*∈*A*. *C*(*x*)) = (⋃*x*∈*B*. *D*(*x*))
⟨*proof*⟩

1.21 Rules for the empty set

lemma *not-mem-empty* [*simp*]: *a* ∉ *0*
⟨*proof*⟩

lemmas *emptyE* [*elim*!] = *not-mem-empty* [*THEN notE*]

lemma *empty-subsetI* [*simp*]: $0 \subseteq A$
⟨*proof*⟩

lemma *equals0I*: $[\![\bigwedge y.\ y{\in}A \Longrightarrow \mathit{False}]\!] \Longrightarrow A{=}0$
⟨*proof*⟩

lemma *equals0D* [*dest*]: $A{=}0 \Longrightarrow a \notin A$
⟨*proof*⟩

declare *sym* [*THEN equals0D*, *dest*]

lemma *not-emptyI*: $a{\in}A \Longrightarrow A \neq 0$
⟨*proof*⟩

lemma *not-emptyE*: $[\![A \neq 0;\ \bigwedge x.\ x{\in}A \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

1.22 Rules for Inter

lemma *Inter-iff*: $A \in \bigcap(C) \longleftrightarrow (\forall\, x{\in}C.\ A{:}\ x) \wedge C{\neq}0$
⟨*proof*⟩

lemma *InterI* [*intro*!]:
$[\![\bigwedge x.\ x{:}\ C \Longrightarrow A{:}\ x;\ \ C{\neq}0]\!] \Longrightarrow A \in \bigcap(C)$
⟨*proof*⟩

lemma *InterD* [*elim*, *Pure.elim*]: $[\![A \in \bigcap(C);\ \ B \in C]\!] \Longrightarrow A \in B$
⟨*proof*⟩

lemma *InterE* [*elim*]:
$[\![A \in \bigcap(C);\ \ B{\notin}C \Longrightarrow R;\ \ A{\in}B \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

1.23 Rules for Intersections of families

lemma *INT-iff*: $b \in (\bigcap x{\in}A.\ B(x)) \longleftrightarrow (\forall\, x{\in}A.\ b \in B(x)) \wedge A{\neq}0$
⟨*proof*⟩

lemma *INT-I*: $[\![\bigwedge x.\ x{:}\ A \Longrightarrow b{:}\ B(x);\ \ A{\neq}0]\!] \Longrightarrow b{:}\ (\bigcap x{\in}A.\ B(x))$
⟨*proof*⟩

lemma *INT-E*: $[\![b \in (\bigcap x{\in}A.\ B(x));\ \ a{:}\ A]\!] \Longrightarrow b \in B(a)$
⟨*proof*⟩

lemma *INT-cong*:
⟦*A*=*B*; ⋀*x*. *x*∈*B* ⟹ *C*(*x*)=*D*(*x*)⟧ ⟹ (⋂*x*∈*A*. *C*(*x*)) = (⋂*x*∈*B*. *D*(*x*))
⟨*proof*⟩

1.24 Rules for Powersets

lemma *PowI*: *A* ⊆ *B* ⟹ *A* ∈ *Pow*(*B*)
⟨*proof*⟩

lemma *PowD*: *A* ∈ *Pow*(*B*) ⟹ *A*⊆*B*
⟨*proof*⟩

declare *Pow-iff* [*iff*]

lemmas *Pow-bottom* = *empty-subsetI* [*THEN PowI*] — *0* ∈ *Pow*(*B*)
lemmas *Pow-top* = *subset-refl* [*THEN PowI*] — *A* ∈ *Pow*(*A*)

1.25 Cantor's Theorem: There is no surjection from a set to its powerset.

lemma *cantor*: ∃ *S* ∈ *Pow*(*A*). ∀ *x*∈*A*. *b*(*x*) ≠ *S*
⟨*proof*⟩

end

2 Unordered Pairs

theory *upair*
imports *ZF-Base*
keywords *print-tcset* :: *diag*
begin

⟨*ML*⟩

lemma *atomize-ball* [*symmetric*, *rulify*]:
(⋀*x*. *x* ∈ *A* ⟹ *P*(*x*)) ≡ *Trueprop* (∀ *x*∈*A*. *P*(*x*))
⟨*proof*⟩

2.1 Unordered Pairs: constant *Upair*

lemma *Upair-iff* [*simp*]: *c* ∈ *Upair*(*a*,*b*) ⟷ (*c*=*a* | *c*=*b*)
⟨*proof*⟩

lemma *UpairI1*: *a* ∈ *Upair*(*a*,*b*)
⟨*proof*⟩

lemma *UpairI2*: *b* ∈ *Upair*(*a*,*b*)
⟨*proof*⟩

lemma *UpairE*: $[\![a \in \mathit{Upair}(b,c);\ \ a{=}b \Longrightarrow P;\ \ a{=}c \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

2.2 Rules for Binary Union, Defined via *Upair*

lemma *Un-iff* [*simp*]: $c \in A \cup B \longleftrightarrow (c \in A \mid c \in B)$
⟨*proof*⟩

lemma *UnI1*: $c \in A \Longrightarrow c \in A \cup B$
⟨*proof*⟩

lemma *UnI2*: $c \in B \Longrightarrow c \in A \cup B$
⟨*proof*⟩

declare *UnI1* [*elim?*] *UnI2* [*elim?*]

lemma *UnE* [*elim*!]: $[\![c \in A \cup B;\ \ c \in A \Longrightarrow P;\ \ c \in B \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *UnE'*: $[\![c \in A \cup B;\ \ c \in A \Longrightarrow P;\ \ [\![c \in B;\ \ c{\notin}A]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *UnCI* [*intro*!]: $(c \notin B \Longrightarrow c \in A) \Longrightarrow c \in A \cup B$
⟨*proof*⟩

2.3 Rules for Binary Intersection, Defined via *Upair*

lemma *Int-iff* [*simp*]: $c \in A \cap B \longleftrightarrow (c \in A \wedge c \in B)$
⟨*proof*⟩

lemma *IntI* [*intro*!]: $[\![c \in A;\ \ c \in B]\!] \Longrightarrow c \in A \cap B$
⟨*proof*⟩

lemma *IntD1*: $c \in A \cap B \Longrightarrow c \in A$
⟨*proof*⟩

lemma *IntD2*: $c \in A \cap B \Longrightarrow c \in B$
⟨*proof*⟩

lemma *IntE* [*elim*!]: $[\![c \in A \cap B;\ [\![c \in A;\ c \in B]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

2.4 Rules for Set Difference, Defined via *Upair*

lemma *Diff-iff* [*simp*]: $c \in A{-}B \longleftrightarrow (c \in A \wedge c{\notin}B)$
⟨*proof*⟩

lemma *DiffI* [*intro*!]: $[\![c \in A;\ \ c \notin B]\!] \Longrightarrow c \in A - B$
⟨*proof*⟩

lemma *DiffD1*: $c \in A - B \Longrightarrow c \in A$
⟨*proof*⟩

lemma *DiffD2*: $c \in A - B \Longrightarrow c \notin B$
⟨*proof*⟩

lemma *DiffE* [*elim*!]: $[\![c \in A - B;\ \ [\![c \in A;\ c \notin B]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

2.5 Rules for *cons*

lemma *cons-iff* [*simp*]: $a \in cons(b,A) \longleftrightarrow (a=b \mid a \in A)$
⟨*proof*⟩

lemma *consI1* [*simp*,*TC*]: $a \in cons(a,B)$
⟨*proof*⟩

lemma *consI2*: $a \in B \Longrightarrow a \in cons(b,B)$
⟨*proof*⟩

lemma *consE* [*elim*!]: $[\![a \in cons(b,A);\ \ a=b \Longrightarrow P;\ \ a \in A \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *consE'*:
$[\![a \in cons(b,A);\ \ a=b \Longrightarrow P;\ \ [\![a \in A;\ \ a \neq b]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *consCI* [*intro*!]: $(a \notin B \Longrightarrow a=b) \Longrightarrow a \in cons(b,B)$
⟨*proof*⟩

lemma *cons-not-0* [*simp*]: $cons(a,B) \neq 0$
⟨*proof*⟩

lemmas *cons-neq-0* = *cons-not-0* [*THEN notE*]

declare *cons-not-0* [*THEN not-sym*, *simp*]

2.6 Singletons

lemma *singleton-iff*: $a \in \{b\} \longleftrightarrow a=b$
⟨*proof*⟩

lemma *singletonI* [*intro*!]: $a \in \{a\}$

⟨*proof*⟩

lemmas *singletonE* = *singleton-iff* [*THEN iffD1*, *elim-format*, *elim*!]

2.7 Descriptions

lemma *the-equality* [*intro*]:
  $[\![P(a);\ \bigwedge x.\ P(x) \Longrightarrow x{=}a]\!] \Longrightarrow (THE\ x.\ P(x)) = a$
⟨*proof*⟩

lemma *the-equality2*: $[\![\exists!x.\ P(x);\ P(a)]\!] \Longrightarrow (THE\ x.\ P(x)) = a$
⟨*proof*⟩

lemma *theI*: $\exists!x.\ P(x) \Longrightarrow P(THE\ x.\ P(x))$
⟨*proof*⟩

lemma *the-0*: $\neg\ (\exists!x.\ P(x)) \Longrightarrow (THE\ x.\ P(x)){=}0$
⟨*proof*⟩

lemma *theI2*:
  assumes *p1*: $\neg\ Q(0) \Longrightarrow \exists!x.\ P(x)$
    and *p2*: $\bigwedge x.\ P(x) \Longrightarrow Q(x)$
  shows $Q(THE\ x.\ P(x))$
⟨*proof*⟩

lemma *the-eq-trivial* [*simp*]: $(THE\ x.\ x = a) = a$
⟨*proof*⟩

lemma *the-eq-trivial2* [*simp*]: $(THE\ x.\ a = x) = a$
⟨*proof*⟩

2.8 Conditional Terms: *if*−*then*−*else*

lemma *if-true* [*simp*]: (*if True then a else b*) = *a*
⟨*proof*⟩

lemma *if-false* [*simp*]: (*if False then a else b*) = *b*
⟨*proof*⟩

lemma *if-cong*:
  $[\![P \longleftrightarrow Q;\ \ Q \Longrightarrow a{=}c;\ \ \neg Q \Longrightarrow b{=}d]\!]$
  $\Longrightarrow$ (*if P then a else b*) = (*if Q then c else d*)
⟨*proof*⟩

lemma *if-weak-cong*: $P \longleftrightarrow Q \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) = (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y)$
⟨*proof*⟩

lemma *if-P*: $P \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ a\ \mathit{else}\ b) = a$
⟨*proof*⟩

lemma *if-not-P*: $\neg P \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ a\ \mathit{else}\ b) = b$
⟨*proof*⟩

lemma *split-if* [*split*]:
$P(\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) \longleftrightarrow ((Q \longrightarrow P(x)) \wedge (\neg Q \longrightarrow P(y)))$
⟨*proof*⟩

lemmas *split-if-eq1* = *split-if* [*of* $\lambda x.\ x = b$] **for** b
lemmas *split-if-eq2* = *split-if* [*of* $\lambda x.\ a = x$] **for** a

lemmas *split-if-mem1* = *split-if* [*of* $\lambda x.\ x \in b$] **for** b
lemmas *split-if-mem2* = *split-if* [*of* $\lambda x.\ a \in x$] **for** a

lemmas *split-ifs* = *split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

lemma *if-iff*: a: $(\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) \longleftrightarrow P \wedge a \in x \mid \neg P \wedge a \in y$
⟨*proof*⟩

lemma *if-type* [*TC*]:
$[\![P \Longrightarrow a \in A;\ \neg P \Longrightarrow b \in A]\!] \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ a\ \mathit{else}\ b)\colon A$
⟨*proof*⟩

lemma *split-if-asm*: $P(\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) \longleftrightarrow (\neg((Q \wedge \neg P(x)) \mid (\neg Q \wedge \neg P(y))))$
⟨*proof*⟩

lemmas *if-splits* = *split-if split-if-asm*

2.9 Consequences of Foundation

lemma *mem-asym*: $[\![a \in b;\ \neg P \Longrightarrow b \in a]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *mem-irrefl*: $a \in a \Longrightarrow P$
⟨*proof*⟩

lemma *mem-not-refl*: $a \notin a$
⟨*proof*⟩

lemma *mem-imp-not-eq*: $a \in A \Longrightarrow a \neq A$
⟨*proof*⟩

lemma *eq-imp-not-mem*: $a{=}A \Longrightarrow a \notin A$
⟨*proof*⟩

2.10 Rules for Successor

lemma *succ-iff*: $i \in succ(j) \longleftrightarrow i{=}j \mid i \in j$
⟨*proof*⟩

lemma *succI1* [*simp*]: $i \in succ(i)$
⟨*proof*⟩

lemma *succI2*: $i \in j \Longrightarrow i \in succ(j)$
⟨*proof*⟩

lemma *succE* [*elim*!]:
$[\![i \in succ(j);\ \ i{=}j \Longrightarrow P;\ \ i \in j \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *succCI* [*intro*!]: $(i{\notin}j \Longrightarrow i{=}j) \Longrightarrow i \in succ(j)$
⟨*proof*⟩

lemma *succ-not-0* [*simp*]: $succ(n) \neq 0$
⟨*proof*⟩

lemmas *succ-neq-0* = *succ-not-0* [*THEN notE*, *elim*!]

declare *succ-not-0* [*THEN not-sym*, *simp*]
declare *sym* [*THEN succ-neq-0*, *elim*!]

lemmas *succ-subsetD* = *succI1* [*THEN* [*2*] *subsetD*]

lemmas *succ-neq-self* = *succI1* [*THEN mem-imp-not-eq*, *THEN not-sym*]

lemma *succ-inject-iff* [*simp*]: $succ(m) = succ(n) \longleftrightarrow m{=}n$
⟨*proof*⟩

lemmas *succ-inject* = *succ-inject-iff* [*THEN iffD1*, *dest*!]

2.11 Miniscoping of the Bounded Universal Quantifier

lemma *ball-simps1*:
$(\forall x{\in}A.\ P(x) \wedge Q) \longleftrightarrow (\forall x{\in}A.\ P(x)) \wedge (A{=}0 \mid Q)$
$(\forall x{\in}A.\ P(x) \mid Q) \longleftrightarrow ((\forall x{\in}A.\ P(x)) \mid Q)$
$(\forall x{\in}A.\ P(x) \longrightarrow Q) \longleftrightarrow ((\exists x{\in}A.\ P(x)) \longrightarrow Q)$
$(\neg(\forall x{\in}A.\ P(x))) \longleftrightarrow (\exists x{\in}A.\ \neg P(x))$
$(\forall x{\in}0.P(x)) \longleftrightarrow \mathit{True}$
$(\forall x{\in}\mathit{succ}(i).P(x)) \longleftrightarrow P(i) \wedge (\forall x{\in}i.\ P(x))$
$(\forall x{\in}\mathit{cons}(a,B).P(x)) \longleftrightarrow P(a) \wedge (\forall x{\in}B.\ P(x))$
$(\forall x{\in}\mathit{RepFun}(A,f).\ P(x)) \longleftrightarrow (\forall y{\in}A.\ P(f(y)))$
$(\forall x{\in}\bigcup(A).P(x)) \longleftrightarrow (\forall y{\in}A.\ \forall x{\in}y.\ P(x))$
⟨*proof*⟩

lemma *ball-simps2*:
$(\forall x{\in}A.\ P \wedge Q(x)) \longleftrightarrow (A{=}0 \mid P) \wedge (\forall x{\in}A.\ Q(x))$
$(\forall x{\in}A.\ P \mid Q(x)) \longleftrightarrow (P \mid (\forall x{\in}A.\ Q(x)))$
$(\forall x{\in}A.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x{\in}A.\ Q(x)))$
⟨*proof*⟩

lemma *ball-simps3*:
$(\forall x{\in}\mathit{Collect}(A,Q).P(x)) \longleftrightarrow (\forall x{\in}A.\ Q(x) \longrightarrow P(x))$
⟨*proof*⟩

lemmas *ball-simps* [*simp*] = *ball-simps1 ball-simps2 ball-simps3*

lemma *ball-conj-distrib*:
$(\forall x{\in}A.\ P(x) \wedge Q(x)) \longleftrightarrow ((\forall x{\in}A.\ P(x)) \wedge (\forall x{\in}A.\ Q(x)))$
⟨*proof*⟩

2.12 Miniscoping of the Bounded Existential Quantifier

lemma *bex-simps1*:
$(\exists x{\in}A.\ P(x) \wedge Q) \longleftrightarrow ((\exists x{\in}A.\ P(x)) \wedge Q)$
$(\exists x{\in}A.\ P(x) \mid Q) \longleftrightarrow (\exists x{\in}A.\ P(x)) \mid (A{\neq}0 \wedge Q)$
$(\exists x{\in}A.\ P(x) \longrightarrow Q) \longleftrightarrow ((\forall x{\in}A.\ P(x)) \longrightarrow (A{\neq}0 \wedge Q))$
$(\exists x{\in}0.P(x)) \longleftrightarrow \mathit{False}$
$(\exists x{\in}\mathit{succ}(i).P(x)) \longleftrightarrow P(i) \mid (\exists x{\in}i.\ P(x))$
$(\exists x{\in}\mathit{cons}(a,B).P(x)) \longleftrightarrow P(a) \mid (\exists x{\in}B.\ P(x))$
$(\exists x{\in}\mathit{RepFun}(A,f).\ P(x)) \longleftrightarrow (\exists y{\in}A.\ P(f(y)))$
$(\exists x{\in}\bigcup(A).P(x)) \longleftrightarrow (\exists y{\in}A.\ \exists x{\in}y.\ \ P(x))$
$(\neg(\exists x{\in}A.\ P(x))) \longleftrightarrow (\forall x{\in}A.\ \neg P(x))$
⟨*proof*⟩

lemma *bex-simps2*:
$(\exists x{\in}A.\ P \wedge Q(x)) \longleftrightarrow (P \wedge (\exists x{\in}A.\ Q(x)))$
$(\exists x{\in}A.\ P \mid Q(x)) \longleftrightarrow (A{\neq}0 \wedge P) \mid (\exists x{\in}A.\ Q(x))$
$(\exists x{\in}A.\ P \longrightarrow Q(x)) \longleftrightarrow ((A{=}0 \mid P) \longrightarrow (\exists x{\in}A.\ Q(x)))$

⟨*proof*⟩

lemma *bex-simps3*:
(∃ *x*∈*Collect*(*A*,*Q*).*P*(*x*)) ⟷ (∃ *x*∈*A*. *Q*(*x*) ∧ *P*(*x*))
⟨*proof*⟩

lemmas *bex-simps* [*simp*] = *bex-simps1 bex-simps2 bex-simps3*

lemma *bex-disj-distrib*:
(∃ *x*∈*A*. *P*(*x*) | *Q*(*x*)) ⟷ ((∃ *x*∈*A*. *P*(*x*)) | (∃ *x*∈*A*. *Q*(*x*)))
⟨*proof*⟩

lemma *bex-triv-one-point1* [*simp*]: (∃ *x*∈*A*. *x*=*a*) ⟷ (*a* ∈ *A*)
⟨*proof*⟩

lemma *bex-triv-one-point2* [*simp*]: (∃ *x*∈*A*. *a*=*x*) ⟷ (*a* ∈ *A*)
⟨*proof*⟩

lemma *bex-one-point1* [*simp*]: (∃ *x*∈*A*. *x*=*a* ∧ *P*(*x*)) ⟷ (*a* ∈ *A* ∧ *P*(*a*))
⟨*proof*⟩

lemma *bex-one-point2* [*simp*]: (∃ *x*∈*A*. *a*=*x* ∧ *P*(*x*)) ⟷ (*a* ∈ *A* ∧ *P*(*a*))
⟨*proof*⟩

lemma *ball-one-point1* [*simp*]: (∀ *x*∈*A*. *x*=*a* ⟶ *P*(*x*)) ⟷ (*a* ∈ *A* ⟶ *P*(*a*))
⟨*proof*⟩

lemma *ball-one-point2* [*simp*]: (∀ *x*∈*A*. *a*=*x* ⟶ *P*(*x*)) ⟷ (*a* ∈ *A* ⟶ *P*(*a*))
⟨*proof*⟩

2.13 Miniscoping of the Replacement Operator

These cover both *Replace* and *Collect*

lemma *Rep-simps* [*simp*]:
{x. y ∈ 0, R(x,y)} = 0
{x ∈ 0. P(x)} = 0
{x ∈ A. Q} = (if Q then A else 0)
RepFun(0,f) = 0
RepFun(succ(i),f) = cons(f(i), RepFun(i,f))
RepFun(cons(a,B),f) = cons(f(a), RepFun(B,f))
⟨*proof*⟩

2.14 Miniscoping of Unions

lemma *UN-simps1*:
(⋃ *x*∈*C*. *cons*(*a*, *B*(*x*))) = (*if C=0 then 0 else cons*(*a*, ⋃ *x*∈*C*. *B*(*x*)))

$(\bigcup x{\in}C.\ A(x) \cup B') = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ (\bigcup x{\in}C.\ A(x)) \cup B')$
$(\bigcup x{\in}C.\ A' \cup B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ A' \cup (\bigcup x{\in}C.\ B(x)))$
$(\bigcup x{\in}C.\ A(x) \cap B') = ((\bigcup x{\in}C.\ A(x)) \cap B')$
$(\bigcup x{\in}C.\ A' \cap B(x)) = (A' \cap (\bigcup x{\in}C.\ B(x)))$
$(\bigcup x{\in}C.\ A(x) - B') = ((\bigcup x{\in}C.\ A(x)) - B')$
$(\bigcup x{\in}C.\ A' - B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ A' - (\bigcap x{\in}C.\ B(x)))$
⟨*proof*⟩

lemma *UN-simps2*:
$(\bigcup x{\in}\bigcup(A).\ B(x)) = (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B(x))$
$(\bigcup z{\in}(\bigcup x{\in}A.\ B(x)).\ C(z)) = (\bigcup x{\in}A.\ \bigcup z{\in}B(x).\ C(z))$
$(\bigcup x{\in}\mathit{RepFun}(A,f).\ B(x)) = (\bigcup a{\in}A.\ B(f(a)))$
⟨*proof*⟩

lemmas *UN-simps* [*simp*] = *UN-simps1 UN-simps2*

Opposite of miniscoping: pull the operator out

lemma *UN-extend-simps1*:
$(\bigcup x{\in}C.\ A(x)) \cup B = (\mathit{if}\ C{=}0\ \mathit{then}\ B\ \mathit{else}\ (\bigcup x{\in}C.\ A(x) \cup B))$
$((\bigcup x{\in}C.\ A(x)) \cap B) = (\bigcup x{\in}C.\ A(x) \cap B)$
$((\bigcup x{\in}C.\ A(x)) - B) = (\bigcup x{\in}C.\ A(x) - B)$
⟨*proof*⟩

lemma *UN-extend-simps2*:
$\mathit{cons}(a, \bigcup x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ \{a\}\ \mathit{else}\ (\bigcup x{\in}C.\ \mathit{cons}(a, B(x))))$
$A \cup (\bigcup x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ A\ \mathit{else}\ (\bigcup x{\in}C.\ A \cup B(x)))$
$(A \cap (\bigcup x{\in}C.\ B(x))) = (\bigcup x{\in}C.\ A \cap B(x))$
$A - (\bigcap x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ A\ \mathit{else}\ (\bigcup x{\in}C.\ A - B(x)))$
$(\bigcup y{\in}A.\ \bigcup x{\in}y.\ B(x)) = (\bigcup x{\in}\bigcup(A).\ B(x))$
$(\bigcup a{\in}A.\ B(f(a))) = (\bigcup x{\in}\mathit{RepFun}(A,f).\ B(x))$
⟨*proof*⟩

lemma *UN-UN-extend*:
$(\bigcup x{\in}A.\ \bigcup z{\in}B(x).\ C(z)) = (\bigcup z{\in}(\bigcup x{\in}A.\ B(x)).\ C(z))$
⟨*proof*⟩

lemmas *UN-extend-simps* = *UN-extend-simps1 UN-extend-simps2 UN-UN-extend*

2.15 Miniscoping of Intersections

lemma *INT-simps1*:
$(\bigcap x{\in}C.\ A(x) \cap B) = (\bigcap x{\in}C.\ A(x)) \cap B$
$(\bigcap x{\in}C.\ A(x) - B) = (\bigcap x{\in}C.\ A(x)) - B$
$(\bigcap x{\in}C.\ A(x) \cup B) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ (\bigcap x{\in}C.\ A(x)) \cup B)$
⟨*proof*⟩

lemma *INT-simps2*:
$(\bigcap x{\in}C.\ A \cap B(x)) = A \cap (\bigcap x{\in}C.\ B(x))$
$(\bigcap x{\in}C.\ A - B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ A - (\bigcup x{\in}C.\ B(x)))$

$(\bigcap x{\in}C.\ cons(a,\ B(x))) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ cons(a,\ \bigcap x{\in}C.\ B(x)))$
$(\bigcap x{\in}C.\ A \cup B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ 0\ \mathit{else}\ A \cup (\bigcap x{\in}C.\ B(x)))$
⟨*proof*⟩

lemmas *INT-simps* [*simp*] = *INT-simps1 INT-simps2*

Opposite of miniscoping: pull the operator out

lemma *INT-extend-simps1*:
$(\bigcap x{\in}C.\ A(x)) \cap B = (\bigcap x{\in}C.\ A(x) \cap B)$
$(\bigcap x{\in}C.\ A(x)) - B = (\bigcap x{\in}C.\ A(x) - B)$
$(\bigcap x{\in}C.\ A(x)) \cup B = (\mathit{if}\ C{=}0\ \mathit{then}\ B\ \mathit{else}\ (\bigcap x{\in}C.\ A(x) \cup B))$
⟨*proof*⟩

lemma *INT-extend-simps2*:
$A \cap (\bigcap x{\in}C.\ B(x)) = (\bigcap x{\in}C.\ A \cap B(x))$
$A - (\bigcup x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ A\ \mathit{else}\ (\bigcap x{\in}C.\ A - B(x)))$
$cons(a, \bigcap x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ \{a\}\ \mathit{else}\ (\bigcap x{\in}C.\ cons(a,\ B(x))))$
$A \cup (\bigcap x{\in}C.\ B(x)) = (\mathit{if}\ C{=}0\ \mathit{then}\ A\ \mathit{else}\ (\bigcap x{\in}C.\ A \cup B(x)))$
⟨*proof*⟩

lemmas *INT-extend-simps* = *INT-extend-simps1 INT-extend-simps2*

2.16 Other simprules

lemma *misc-simps* [*simp*]:
$0 \cup A = A$
$A \cup 0 = A$
$0 \cap A = 0$
$A \cap 0 = 0$
$0 - A = 0$
$A - 0 = A$
$\bigcup(0) = 0$
$\bigcup(cons(b,A)) = b \cup \bigcup(A)$
$\bigcap(\{b\}) = b$
⟨*proof*⟩

end

3 Ordered Pairs

theory *pair* **imports** *upair*
begin

⟨*ML*⟩

lemma *singleton-eq-iff* [*iff*]: $\{a\} = \{b\} \longleftrightarrow a{=}b$

⟨*proof*⟩

lemma *doubleton-eq-iff*: $\{a,b\} = \{c,d\} \longleftrightarrow (a{=}c \wedge b{=}d) \mid (a{=}d \wedge b{=}c)$
⟨*proof*⟩

lemma *Pair-iff* [*simp*]: $\langle a,b\rangle = \langle c,d\rangle \longleftrightarrow a{=}c \wedge b{=}d$
⟨*proof*⟩

lemmas *Pair-inject* = *Pair-iff* [*THEN iffD1*, *THEN conjE*, *elim*!]

lemmas *Pair-inject1* = *Pair-iff* [*THEN iffD1*, *THEN conjunct1*]
lemmas *Pair-inject2* = *Pair-iff* [*THEN iffD1*, *THEN conjunct2*]

lemma *Pair-not-0*: $\langle a,b\rangle \neq 0$
⟨*proof*⟩

lemmas *Pair-neq-0* = *Pair-not-0* [*THEN notE*, *elim*!]

declare *sym* [*THEN Pair-neq-0*, *elim*!]

lemma *Pair-neq-fst*: $\langle a,b\rangle{=}a \Longrightarrow P$
⟨*proof*⟩

lemma *Pair-neq-snd*: $\langle a,b\rangle{=}b \Longrightarrow P$
⟨*proof*⟩

3.1 Sigma: Disjoint Union of a Family of Sets

Generalizes Cartesian product

lemma *Sigma-iff* [*simp*]: $\langle a,b\rangle$: $Sigma(A,B) \longleftrightarrow a \in A \wedge b \in B(a)$
⟨*proof*⟩

lemma *SigmaI* [*TC*,*intro*!]: $[\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow \langle a,b\rangle \in Sigma(A,B)$
⟨*proof*⟩

lemmas *SigmaD1* = *Sigma-iff* [*THEN iffD1*, *THEN conjunct1*]
lemmas *SigmaD2* = *Sigma-iff* [*THEN iffD1*, *THEN conjunct2*]

lemma *SigmaE* [*elim*!]:
$[\![c \in Sigma(A,B);$
$\bigwedge x\ y.[\![x \in A;\ \ y \in B(x);\ \ c{=}\langle x,y\rangle]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *SigmaE2* [*elim*!]:
$[\![\langle a,b\rangle \in Sigma(A,B);$
$[\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$

⟨*proof*⟩

lemma *Sigma-cong*:
⟦A=A'; ⋀x. $x \in A' \Longrightarrow B(x)$=$B'(x)$⟧ ⟹
Sigma(A,B) = *Sigma*(A',B')
⟨*proof*⟩

lemma *Sigma-empty1* [*simp*]: *Sigma*(*0*,B) = *0*
⟨*proof*⟩

lemma *Sigma-empty2* [*simp*]: A∗*0* = *0*
⟨*proof*⟩

lemma *Sigma-empty-iff*: A∗B=*0* ⟷ A=*0* | B=*0*
⟨*proof*⟩

3.2 Projections *fst* and *snd*

lemma *fst-conv* [*simp*]: *fst*(⟨a,b⟩) = a
⟨*proof*⟩

lemma *snd-conv* [*simp*]: *snd*(⟨a,b⟩) = b
⟨*proof*⟩

lemma *fst-type* [*TC*]: $p \in$ *Sigma*(A,B) ⟹ *fst*(p) ∈ A
⟨*proof*⟩

lemma *snd-type* [*TC*]: $p \in$ *Sigma*(A,B) ⟹ *snd*(p) ∈ B(*fst*(p))
⟨*proof*⟩

lemma *Pair-fst-snd-eq*: $a \in$ *Sigma*(A,B) ⟹ <*fst*(a),*snd*(a)> = a
⟨*proof*⟩

3.3 The Eliminator, *split*

lemma *split* [*simp*]: *split*(λx y. c(x,y), ⟨a,b⟩) ≡ c(a,b)
⟨*proof*⟩

lemma *split-type* [*TC*]:
⟦$p \in$ *Sigma*(A,B);
⋀x y.⟦$x \in A$; $y \in B(x)$⟧ ⟹ c(x,y):C(⟨x,y⟩)
⟧ ⟹ *split*(λx y. c(x,y), p) ∈ C(p)
⟨*proof*⟩

lemma *expand-split*:
$u \in A$∗B ⟹
R(*split*(c,u)) ⟷ (∀ x∈A. ∀ y∈B. u = ⟨x,y⟩ ⟶ R(c(x,y)))
⟨*proof*⟩

3.4 A version of *split* for Formulae: Result Type *o*

lemma *splitI*: $R(a,b) \Longrightarrow split(R, \langle a,b\rangle)$
⟨*proof*⟩

lemma *splitE*:
$[\![split(R,z);\ z \in Sigma(A,B);$
$\bigwedge x\ y.\ [\![z = \langle x,y\rangle;\ R(x,y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *splitD*: $split(R,\langle a,b\rangle) \Longrightarrow R(a,b)$
⟨*proof*⟩

Complex rules for Sigma.

lemma *split-paired-Bex-Sigma* [*simp*]:
$(\exists\, z \in Sigma(A,B).\ P(z)) \longleftrightarrow (\exists\, x \in A.\ \exists\, y \in B(x).\ P(\langle x,y\rangle))$
⟨*proof*⟩

lemma *split-paired-Ball-Sigma* [*simp*]:
$(\forall\, z \in Sigma(A,B).\ P(z)) \longleftrightarrow (\forall\, x \in A.\ \forall\, y \in B(x).\ P(\langle x,y\rangle))$
⟨*proof*⟩

end

4 Basic Equalities and Inclusions

theory *equalities* **imports** *pair* **begin**

These cover union, intersection, converse, domain, range, etc. Philippe de Groote proved many of the inclusions.

lemma *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$
⟨*proof*⟩

lemma *the-eq-0* [*simp*]: $(THE\ x.\ False) = 0$
⟨*proof*⟩

4.1 Bounded Quantifiers

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall\, x \in A \cup B.\ P(x)) \longleftrightarrow (\forall\, x \in A.\ P(x)) \wedge (\forall\, x \in B.\ P(x))$
⟨*proof*⟩

lemma *bex-Un*: $(\exists\, x \in A \cup B.\ P(x)) \longleftrightarrow (\exists\, x \in A.\ P(x)) \mid (\exists\, x \in B.\ P(x))$
⟨*proof*⟩

lemma *ball-UN*: $(\forall z \in (\bigcup x{\in}A.\ B(x)).\ P(z)) \longleftrightarrow (\forall x{\in}A.\ \forall z \in B(x).\ P(z))$
⟨*proof*⟩

lemma *bex-UN*: $(\exists z \in (\bigcup x{\in}A.\ B(x)).\ P(z)) \longleftrightarrow (\exists x{\in}A.\ \exists z{\in}B(x).\ P(z))$
⟨*proof*⟩

4.2 Converse of a Relation

lemma *converse-iff* [*simp*]: $\langle a,b\rangle \in converse(r) \longleftrightarrow \langle b,a\rangle \in r$
⟨*proof*⟩

lemma *converseI* [*intro*!]: $\langle a,b\rangle \in r \Longrightarrow \langle b,a\rangle \in converse(r)$
⟨*proof*⟩

lemma *converseD*: $\langle a,b\rangle \in converse(r) \Longrightarrow \langle b,a\rangle \in r$
⟨*proof*⟩

lemma *converseE* [*elim*!]:
$[\![yx \in converse(r);$
$\bigwedge x\ y.\ [\![yx=\langle y,x\rangle;\ \langle x,y\rangle \in r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
⟨*proof*⟩

lemma *converse-converse*: $r \subseteq Sigma(A,B) \Longrightarrow converse(converse(r)) = r$
⟨*proof*⟩

lemma *converse-type*: $r \subseteq A*B \Longrightarrow converse(r) \subseteq B*A$
⟨*proof*⟩

lemma *converse-prod* [*simp*]: $converse(A*B) = B*A$
⟨*proof*⟩

lemma *converse-empty* [*simp*]: $converse(0) = 0$
⟨*proof*⟩

lemma *converse-subset-iff*:
$A \subseteq Sigma(X,Y) \Longrightarrow converse(A) \subseteq converse(B) \longleftrightarrow A \subseteq B$
⟨*proof*⟩

4.3 Finite Set Constructions Using *cons*

lemma *cons-subsetI*: $[\![a{\in}C;\ B{\subseteq}C]\!] \Longrightarrow cons(a,B) \subseteq C$
⟨*proof*⟩

lemma *subset-consI*: $B \subseteq cons(a,B)$
⟨*proof*⟩

lemma *cons-subset-iff* [*iff*]: $cons(a,B){\subseteq}C \longleftrightarrow a{\in}C \wedge B{\subseteq}C$
⟨*proof*⟩

lemmas *cons-subsetE* = *cons-subset-iff* [*THEN iffD1*, *THEN conjE*]

lemma *subset-empty-iff*: $A \subseteq 0 \longleftrightarrow A = 0$
⟨*proof*⟩

lemma *subset-cons-iff*: $C \subseteq cons(a,B) \longleftrightarrow C \subseteq B \mid (a \in C \wedge C - \{a\} \subseteq B)$
⟨*proof*⟩

lemma *cons-eq*: $\{a\} \cup B = cons(a,B)$
⟨*proof*⟩

lemma *cons-commute*: $cons(a, cons(b, C)) = cons(b, cons(a, C))$
⟨*proof*⟩

lemma *cons-absorb*: $a \colon B \Longrightarrow cons(a,B) = B$
⟨*proof*⟩

lemma *cons-Diff*: $a \colon B \Longrightarrow cons(a, B - \{a\}) = B$
⟨*proof*⟩

lemma *Diff-cons-eq*: $cons(a,B) - C = (\textit{if } a \in C \textit{ then } B - C \textit{ else } cons(a, B - C))$
⟨*proof*⟩

lemma *equal-singleton*: $[\![a \colon C;\ \bigwedge y.\ y \in C \Longrightarrow y = b]\!] \Longrightarrow C = \{b\}$
⟨*proof*⟩

lemma [*simp*]: $cons(a, cons(a,B)) = cons(a,B)$
⟨*proof*⟩

lemma *singleton-subsetI*: $a \in C \Longrightarrow \{a\} \subseteq C$
⟨*proof*⟩

lemma *singleton-subsetD*: $\{a\} \subseteq C \Longrightarrow a \in C$
⟨*proof*⟩

lemma *subset-succI*: $i \subseteq succ(i)$
⟨*proof*⟩

lemma *succ-subsetI*: $[\![i \in j;\ i \subseteq j]\!] \Longrightarrow succ(i) \subseteq j$
⟨*proof*⟩

lemma *succ-subsetE*:
$\quad [\![succ(i) \subseteq j;\ \ [\![i \in j;\ \ i \subseteq j]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *succ-subset-iff*: $succ(a) \subseteq B \longleftrightarrow (a \subseteq B \wedge a \in B)$
⟨*proof*⟩

4.4 Binary Intersection

lemma *Int-subset-iff*: $C \subseteq A \cap B \longleftrightarrow C \subseteq A \wedge C \subseteq B$
⟨*proof*⟩

lemma *Int-lower1*: $A \cap B \subseteq A$
⟨*proof*⟩

lemma *Int-lower2*: $A \cap B \subseteq B$
⟨*proof*⟩

lemma *Int-greatest*: $[\![C \subseteq A;\ \ C \subseteq B]\!] \Longrightarrow C \subseteq A \cap B$
⟨*proof*⟩

lemma *Int-cons*: $cons(a,B) \cap C \subseteq cons(a,\ B \cap C)$
⟨*proof*⟩

lemma *Int-absorb* [*simp*]: $A \cap A = A$
⟨*proof*⟩

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
⟨*proof*⟩

lemma *Int-commute*: $A \cap B = B \cap A$
⟨*proof*⟩

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
⟨*proof*⟩

lemma *Int-assoc*: $(A \cap B) \cap C \ = \ A \cap (B \cap C)$
⟨*proof*⟩

lemmas *Int-ac= Int-assoc Int-left-absorb Int-commute Int-left-commute*

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
⟨*proof*⟩

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
⟨*proof*⟩

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
⟨*proof*⟩

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
⟨*proof*⟩

lemma *subset-Int-iff*: $A \subseteq B \longleftrightarrow A \cap B = A$
⟨*proof*⟩

lemma *subset-Int-iff2*: $A \subseteq B \longleftrightarrow B \cap A = A$
⟨*proof*⟩

lemma *Int-Diff-eq*: $C \subseteq A \Longrightarrow (A - B) \cap C = C - B$
⟨*proof*⟩

lemma *Int-cons-left*:
$cons(a,A) \cap B = (\textit{if } a \in B \textit{ then } cons(a, A \cap B) \textit{ else } A \cap B)$
⟨*proof*⟩

lemma *Int-cons-right*:
$A \cap cons(a, B) = (\textit{if } a \in A \textit{ then } cons(a, A \cap B) \textit{ else } A \cap B)$
⟨*proof*⟩

lemma *cons-Int-distrib*: $cons(x, A \cap B) = cons(x, A) \cap cons(x, B)$
⟨*proof*⟩

4.5 Binary Union

lemma *Un-subset-iff*: $A \cup B \subseteq C \longleftrightarrow A \subseteq C \wedge B \subseteq C$
⟨*proof*⟩

lemma *Un-upper1*: $A \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-upper2*: $B \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-least*: $[\![A \subseteq C;\ \ B \subseteq C]\!] \Longrightarrow A \cup B \subseteq C$
⟨*proof*⟩

lemma *Un-cons*: $cons(a,B) \cup C = cons(a, B \cup C)$
⟨*proof*⟩

lemma *Un-absorb* [*simp*]: $A \cup A = A$
⟨*proof*⟩

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
⟨*proof*⟩

lemma *Un-commute*: $A \cup B = B \cup A$
⟨*proof*⟩

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
⟨*proof*⟩

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
⟨*proof*⟩

lemmas *Un-ac = Un-assoc Un-left-absorb Un-commute Un-left-commute*

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
⟨*proof*⟩

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
⟨*proof*⟩

lemma *Un-Int-distrib*: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
⟨*proof*⟩

lemma *subset-Un-iff*: $A \subseteq B \longleftrightarrow A \cup B = B$
⟨*proof*⟩

lemma *subset-Un-iff2*: $A \subseteq B \longleftrightarrow B \cup A = B$
⟨*proof*⟩

lemma *Un-empty* [*iff*]: $(A \cup B = 0) \longleftrightarrow (A = 0 \wedge B = 0)$
⟨*proof*⟩

lemma *Un-eq-Union*: $A \cup B = \bigcup(\{A, B\})$
⟨*proof*⟩

4.6 Set Difference

lemma *Diff-subset*: $A - B \subseteq A$
⟨*proof*⟩

lemma *Diff-contains*: $[\![C \subseteq A;\ C \cap B = 0]\!] \Longrightarrow C \subseteq A - B$
⟨*proof*⟩

lemma *subset-Diff-cons-iff*: $B \subseteq A - cons(c,C) \longleftrightarrow B \subseteq A - C \wedge c \notin B$
⟨*proof*⟩

lemma *Diff-cancel*: $A - A = 0$
⟨*proof*⟩

lemma *Diff-triv*: $A \cap B = 0 \Longrightarrow A - B = A$
⟨*proof*⟩

lemma *empty-Diff* [*simp*]: $0 - A = 0$
⟨*proof*⟩

lemma *Diff-0* [*simp*]: $A - 0 = A$
⟨*proof*⟩

lemma *Diff-eq-0-iff*: $A - B = 0 \longleftrightarrow A \subseteq B$
⟨*proof*⟩

lemma *Diff-cons*: $A - cons(a,B) = A - B - \{a\}$
⟨*proof*⟩

lemma *Diff-cons2*: $A - cons(a,B) = A - \{a\} - B$
⟨*proof*⟩

lemma *Diff-disjoint*: $A \cap (B-A) = 0$
⟨*proof*⟩

lemma *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B-A) = B$
⟨*proof*⟩

lemma *subset-Un-Diff*: $A \subseteq B \cup (A - B)$
⟨*proof*⟩

lemma *double-complement*: $[\![A \subseteq B;\ B \subseteq C]\!] \Longrightarrow B-(C-A) = A$
⟨*proof*⟩

lemma *double-complement-Un*: $(A \cup B) - (B-A) = A$
⟨*proof*⟩

lemma *Un-Int-crazy*:
$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
⟨*proof*⟩

lemma *Diff-Un*: $A - (B \cup C) = (A-B) \cap (A-C)$
⟨*proof*⟩

lemma *Diff-Int*: $A - (B \cap C) = (A-B) \cup (A-C)$
⟨*proof*⟩

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
⟨*proof*⟩

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
⟨*proof*⟩

lemma *Diff-Int-distrib*: $C \cap (A-B) = (C \cap A) - (C \cap B)$
⟨*proof*⟩

lemma *Diff-Int-distrib2*: $(A-B) \cap C = (A \cap C) - (B \cap C)$
⟨*proof*⟩

lemma *Un-Int-assoc-iff*: $(A \cap B) \cup C = A \cap (B \cup C) \longleftrightarrow C \subseteq A$
⟨*proof*⟩

4.7 Big Union and Intersection

lemma *Union-subset-iff*: $\bigcup(A) \subseteq C \longleftrightarrow (\forall x \in A.\ x \subseteq C)$
⟨*proof*⟩

lemma *Union-upper*: $B \in A \Longrightarrow B \subseteq \bigcup(A)$
⟨*proof*⟩

lemma *Union-least*: $[\![\bigwedge x.\ x \in A \Longrightarrow x \subseteq C]\!] \Longrightarrow \bigcup(A) \subseteq C$
⟨*proof*⟩

lemma *Union-cons* [*simp*]: $\bigcup(cons(a,B)) = a \cup \bigcup(B)$
⟨*proof*⟩

lemma *Union-Un-distrib*: $\bigcup(A \cup B) = \bigcup(A) \cup \bigcup(B)$
⟨*proof*⟩

lemma *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup(A) \cap \bigcup(B)$
⟨*proof*⟩

lemma *Union-disjoint*: $\bigcup(C) \cap A = 0 \longleftrightarrow (\forall B \in C.\ B \cap A = 0)$
⟨*proof*⟩

lemma *Union-empty-iff*: $\bigcup(A) = 0 \longleftrightarrow (\forall B \in A.\ B=0)$
⟨*proof*⟩

lemma *Int-Union2*: $\bigcup(B) \cap A = (\bigcup C \in B.\ C \cap A)$
⟨*proof*⟩

lemma *Inter-subset-iff*: $A \neq 0 \implies C \subseteq \bigcap(A) \longleftrightarrow (\forall x \in A.\ C \subseteq x)$
⟨*proof*⟩

lemma *Inter-lower*: $B \in A \Longrightarrow \bigcap(A) \subseteq B$
⟨*proof*⟩

lemma *Inter-greatest*: $[\![A \neq 0;\ \bigwedge x.\ x \in A \Longrightarrow C \subseteq x]\!] \Longrightarrow C \subseteq \bigcap(A)$
⟨*proof*⟩

lemma *INT-lower*: $x{\in}A \Longrightarrow (\bigcap x{\in}A.\ B(x)) \subseteq B(x)$
⟨*proof*⟩

lemma *INT-greatest*: $[\![A{\neq}0;\ \bigwedge x.\ x{\in}A \Longrightarrow C{\subseteq}B(x)]\!] \Longrightarrow C \subseteq (\bigcap x{\in}A.\ B(x))$
⟨*proof*⟩

lemma *Inter-0* [*simp*]: $\bigcap(0) = 0$
⟨*proof*⟩

lemma *Inter-Un-subset*:
$[\![z{\in}A;\ z{\in}B]\!] \Longrightarrow \bigcap(A) \cup \bigcap(B) \subseteq \bigcap(A \cap B)$
⟨*proof*⟩

lemma *Inter-Un-distrib*:
$[\![A{\neq}0;\ \ B{\neq}0]\!] \Longrightarrow \bigcap(A \cup B) = \bigcap(A) \cap \bigcap(B)$
⟨*proof*⟩

lemma *Union-singleton*: $\bigcup(\{b\}) = b$
⟨*proof*⟩

lemma *Inter-singleton*: $\bigcap(\{b\}) = b$
⟨*proof*⟩

lemma *Inter-cons* [*simp*]:
$\bigcap(cons(a,B)) = (\mathit{if}\ B{=}0\ \mathit{then}\ a\ \mathit{else}\ a \cap \bigcap(B))$
⟨*proof*⟩

4.8 Unions and Intersections of Families

lemma *subset-UN-iff-eq*: $A \subseteq (\bigcup i{\in}I.\ B(i)) \longleftrightarrow A = (\bigcup i{\in}I.\ A \cap B(i))$
⟨*proof*⟩

lemma *UN-subset-iff*: $(\bigcup x{\in}A.\ B(x)) \subseteq C \longleftrightarrow (\forall x{\in}A.\ B(x) \subseteq C)$
⟨*proof*⟩

lemma *UN-upper*: $x{\in}A \Longrightarrow B(x) \subseteq (\bigcup x{\in}A.\ B(x))$
⟨*proof*⟩

lemma *UN-least*: $[\![\bigwedge x.\ x{\in}A \Longrightarrow B(x){\subseteq}C]\!] \Longrightarrow (\bigcup x{\in}A.\ B(x)) \subseteq C$
⟨*proof*⟩

lemma *Union-eq-UN*: $\bigcup(A) = (\bigcup x{\in}A.\ x)$
⟨*proof*⟩

lemma *Inter-eq-INT*: $\bigcap(A) = (\bigcap x{\in}A.\ x)$

⟨*proof*⟩

lemma *UN-0* [*simp*]: $(\bigcup i{\in}0.\ A(i)) = 0$
⟨*proof*⟩

lemma *UN-singleton*: $(\bigcup x{\in}A.\ \{x\}) = A$
⟨*proof*⟩

lemma *UN-Un*: $(\bigcup i{\in} A \cup B.\ C(i)) = (\bigcup i{\in} A.\ C(i)) \cup (\bigcup i{\in}B.\ C(i))$
⟨*proof*⟩

lemma *INT-Un*: $(\bigcap i{\in}I \cup J.\ A(i)) =$
$\quad$ (*if* $I{=}0$ *then* $\bigcap j{\in}J.\ A(j)$
$\quad\quad$ *else if* $J{=}0$ *then* $\bigcap i{\in}I.\ A(i)$
$\quad\quad$ *else* $((\bigcap i{\in}I.\ A(i)) \cap\ (\bigcap j{\in}J.\ A(j))))$
⟨*proof*⟩

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B(y)).\ C(x)) = (\bigcup y{\in}A.\ \bigcup x{\in} B(y).\ C(x))$
⟨*proof*⟩

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A(i)) = (\bigcup i{\in}I.\ B \cap A(i))$
⟨*proof*⟩

lemma *Un-INT-distrib*: $I{\neq}0 \Longrightarrow B \cup (\bigcap i{\in}I.\ A(i)) = (\bigcap i{\in}I.\ B \cup A(i))$
⟨*proof*⟩

lemma *Int-UN-distrib2*:
$\quad (\bigcup i{\in}I.\ A(i)) \cap (\bigcup j{\in}J.\ B(j)) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A(i) \cap B(j))$
⟨*proof*⟩

lemma *Un-INT-distrib2*: $[\![I{\neq}0;\ \ J{\neq}0]\!] \Longrightarrow$
$\quad (\bigcap i{\in}I.\ A(i)) \cup (\bigcap j{\in}J.\ B(j)) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A(i) \cup B(j))$
⟨*proof*⟩

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (\textit{if } A{=}0 \textit{ then } 0 \textit{ else } c)$
⟨*proof*⟩

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (\textit{if } A{=}0 \textit{ then } 0 \textit{ else } c)$
⟨*proof*⟩

lemma *UN-RepFun* [*simp*]: $(\bigcup y{\in}\ \textit{RepFun}(A,f).\ B(y)) = (\bigcup x{\in}A.\ B(f(x)))$
⟨*proof*⟩

lemma *INT-RepFun* [*simp*]: $(\bigcap x{\in}\textit{RepFun}(A,f).\ B(x)) \quad = (\bigcap a{\in}A.\ B(f(a)))$
⟨*proof*⟩

lemma *INT-Union-eq*:
$\quad 0 \notin A \Longrightarrow (\bigcap x{\in}\bigcup(A).\ B(x)) = (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B(x))$

⟨*proof*⟩

lemma *INT-UN-eq*:
(∀ *x*∈*A*. *B*(*x*) ≠ *0*)
⟹ (⋂ *z*∈ (⋃ *x*∈*A*. *B*(*x*)). *C*(*z*)) = (⋂ *x*∈*A*. ⋂ *z*∈ *B*(*x*). *C*(*z*))
⟨*proof*⟩

lemma *UN-Un-distrib*:
(⋃ *i*∈*I*. *A*(*i*) ∪ *B*(*i*)) = (⋃ *i*∈*I*. *A*(*i*)) ∪ (⋃ *i*∈*I*. *B*(*i*))
⟨*proof*⟩

lemma *INT-Int-distrib*:
I≠*0* ⟹ (⋂ *i*∈*I*. *A*(*i*) ∩ *B*(*i*)) = (⋂ *i*∈*I*. *A*(*i*)) ∩ (⋂ *i*∈*I*. *B*(*i*))
⟨*proof*⟩

lemma *UN-Int-subset*:
(⋃ *z*∈*I* ∩ *J*. *A*(*z*)) ⊆ (⋃ *z*∈*I*. *A*(*z*)) ∩ (⋃ *z*∈*J*. *A*(*z*))
⟨*proof*⟩

lemma *Diff-UN*: *I*≠*0* ⟹ *B* − (⋃ *i*∈*I*. *A*(*i*)) = (⋂ *i*∈*I*. *B* − *A*(*i*))
⟨*proof*⟩

lemma *Diff-INT*: *I*≠*0* ⟹ *B* − (⋂ *i*∈*I*. *A*(*i*)) = (⋃ *i*∈*I*. *B* − *A*(*i*))
⟨*proof*⟩

lemma *Sigma-cons1*: *Sigma*(*cons*(*a*,*B*), *C*) = ({*a*}∗*C*(*a*)) ∪ *Sigma*(*B*,*C*)
⟨*proof*⟩

lemma *Sigma-cons2*: *A* ∗ *cons*(*b*,*B*) = *A*∗{*b*} ∪ *A*∗*B*
⟨*proof*⟩

lemma *Sigma-succ1*: *Sigma*(*succ*(*A*), *B*) = ({*A*}∗*B*(*A*)) ∪ *Sigma*(*A*,*B*)
⟨*proof*⟩

lemma *Sigma-succ2*: *A* ∗ *succ*(*B*) = *A*∗{*B*} ∪ *A*∗*B*
⟨*proof*⟩

lemma *SUM-UN-distrib1*:
(∑ *x* ∈ (⋃ *y*∈*A*. *C*(*y*)). *B*(*x*)) = (⋃ *y*∈*A*. ∑ *x*∈*C*(*y*). *B*(*x*))

⟨*proof*⟩

lemma *SUM-UN-distrib2*:
$(\sum i{\in}I.\ \bigcup j{\in}J.\ C(i,j)) = (\bigcup j{\in}J.\ \sum i{\in}I.\ C(i,j))$
⟨*proof*⟩

lemma *SUM-Un-distrib1*:
$(\sum i{\in}I \cup J.\ C(i)) = (\sum i{\in}I.\ C(i)) \cup (\sum j{\in}J.\ C(j))$
⟨*proof*⟩

lemma *SUM-Un-distrib2*:
$(\sum i{\in}I.\ A(i) \cup B(i)) = (\sum i{\in}I.\ A(i)) \cup (\sum i{\in}I.\ B(i))$
⟨*proof*⟩

lemma *prod-Un-distrib2*: $I * (A \cup B) = I{*}A \cup I{*}B$
⟨*proof*⟩

lemma *SUM-Int-distrib1*:
$(\sum i{\in}I \cap J.\ C(i)) = (\sum i{\in}I.\ C(i)) \cap (\sum j{\in}J.\ C(j))$
⟨*proof*⟩

lemma *SUM-Int-distrib2*:
$(\sum i{\in}I.\ A(i) \cap B(i)) = (\sum i{\in}I.\ A(i)) \cap (\sum i{\in}I.\ B(i))$
⟨*proof*⟩

lemma *prod-Int-distrib2*: $I * (A \cap B) = I{*}A \cap I{*}B$
⟨*proof*⟩

lemma *SUM-eq-UN*: $(\sum i{\in}I.\ A(i)) = (\bigcup i{\in}I.\ \{i\} * A(i))$
⟨*proof*⟩

lemma *times-subset-iff*:
$(A'{*}B' \subseteq A{*}B) \longleftrightarrow (A' = 0 \mid B' = 0 \mid (A'{\subseteq}A) \wedge (B'{\subseteq}B))$
⟨*proof*⟩

lemma *Int-Sigma-eq*:
$(\sum x \in A'.\ B'(x)) \cap (\sum x \in A.\ B(x)) = (\sum x \in A' \cap A.\ B'(x) \cap B(x))$
⟨*proof*⟩

lemma *domain-iff*: *a*: *domain*$(r) \longleftrightarrow (\exists\, y.\ \langle a,y\rangle{\in}\ r)$
⟨*proof*⟩

lemma *domainI* [*intro*]: $\langle a,b\rangle{\in}\ r \Longrightarrow a$: *domain*$(r)$
⟨*proof*⟩

lemma *domainE* [*elim*!]:
$[\![a \in domain(r);\ \bigwedge y.\ \langle a,y\rangle \in r \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *domain-subset*: $domain(Sigma(A,B)) \subseteq A$
⟨*proof*⟩

lemma *domain-of-prod*: $b \in B \Longrightarrow domain(A*B) = A$
⟨*proof*⟩

lemma *domain-0* [*simp*]: $domain(0) = 0$
⟨*proof*⟩

lemma *domain-cons* [*simp*]: $domain(cons(\langle a,b\rangle,r)) = cons(a,\ domain(r))$
⟨*proof*⟩

lemma *domain-Un-eq* [*simp*]: $domain(A \cup B) = domain(A) \cup domain(B)$
⟨*proof*⟩

lemma *domain-Int-subset*: $domain(A \cap B) \subseteq domain(A) \cap domain(B)$
⟨*proof*⟩

lemma *domain-Diff-subset*: $domain(A) - domain(B) \subseteq domain(A - B)$
⟨*proof*⟩

lemma *domain-UN*: $domain(\bigcup x{\in}A.\ B(x)) = (\bigcup x{\in}A.\ domain(B(x)))$
⟨*proof*⟩

lemma *domain-Union*: $domain(\bigcup(A)) = (\bigcup x{\in}A.\ domain(x))$
⟨*proof*⟩

lemma *rangeI* [*intro*]: $\langle a,b\rangle \in r \Longrightarrow b \in range(r)$
⟨*proof*⟩

lemma *rangeE* [*elim*!]: $[\![b \in range(r);\ \bigwedge x.\ \langle x,b\rangle \in r \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *range-subset*: $range(A*B) \subseteq B$
⟨*proof*⟩

lemma *range-of-prod*: $a \in A \Longrightarrow range(A*B) = B$
⟨*proof*⟩

lemma *range-0* [*simp*]: $range(0) = 0$
⟨*proof*⟩

lemma *range-cons* [*simp*]: $range(cons(\langle a,b\rangle,r)) = cons(b,\ range(r))$
⟨*proof*⟩

lemma *range-Un-eq* [*simp*]: $range(A \cup B) = range(A) \cup range(B)$
⟨*proof*⟩

lemma *range-Int-subset*: $range(A \cap B) \subseteq range(A) \cap range(B)$
⟨*proof*⟩

lemma *range-Diff-subset*: $range(A) - range(B) \subseteq range(A - B)$
⟨*proof*⟩

lemma *domain-converse* [*simp*]: $domain(converse(r)) = range(r)$
⟨*proof*⟩

lemma *range-converse* [*simp*]: $range(converse(r)) = domain(r)$
⟨*proof*⟩

lemma *fieldI1*: $\langle a,b\rangle \in r \Longrightarrow a \in field(r)$
⟨*proof*⟩

lemma *fieldI2*: $\langle a,b\rangle \in r \Longrightarrow b \in field(r)$
⟨*proof*⟩

lemma *fieldCI* [*intro*]:
$(\neg \langle c,a\rangle \in r \Longrightarrow \langle a,b\rangle \in r) \Longrightarrow a \in field(r)$
⟨*proof*⟩

lemma *fieldE* [*elim*!]:
$[\![a \in field(r);$
$\bigwedge x.\ \langle a,x\rangle \in r \Longrightarrow P;$
$\bigwedge x.\ \langle x,a\rangle \in r \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *field-subset*: $field(A*B) \subseteq A \cup B$
⟨*proof*⟩

lemma *domain-subset-field*: $domain(r) \subseteq field(r)$
⟨*proof*⟩

lemma *range-subset-field*: $range(r) \subseteq field(r)$
⟨*proof*⟩

lemma *domain-times-range*: $r \subseteq Sigma(A,B) \Longrightarrow r \subseteq domain(r)*range(r)$
⟨*proof*⟩

lemma *field-times-field*: $r \subseteq Sigma(A,B) \Longrightarrow r \subseteq field(r)*field(r)$
⟨*proof*⟩

lemma *relation-field-times-field*: $relation(r) \Longrightarrow r \subseteq field(r)*field(r)$
⟨*proof*⟩

lemma *field-of-prod*: $field(A*A) = A$
⟨*proof*⟩

lemma *field-0* [*simp*]: $field(0) = 0$
⟨*proof*⟩

lemma *field-cons* [*simp*]: $field(cons(\langle a,b\rangle,r)) = cons(a,\ cons(b,\ field(r)))$
⟨*proof*⟩

lemma *field-Un-eq* [*simp*]: $field(A \cup B) = field(A) \cup field(B)$
⟨*proof*⟩

lemma *field-Int-subset*: $field(A \cap B) \subseteq field(A) \cap field(B)$
⟨*proof*⟩

lemma *field-Diff-subset*: $field(A) - field(B) \subseteq field(A - B)$
⟨*proof*⟩

lemma *field-converse* [*simp*]: $field(converse(r)) = field(r)$
⟨*proof*⟩

lemma *rel-Union*: $(\forall\, x{\in}S.\ \exists\, A\ B.\ x \subseteq A*B) \Longrightarrow$
$\bigcup(S) \subseteq domain(\bigcup(S)) * range(\bigcup(S))$
⟨*proof*⟩

lemma *rel-Un*: $[\![r \subseteq A*B;\ \ s \subseteq C*D]\!] \Longrightarrow (r \cup s) \subseteq (A \cup C) * (B \cup D)$
⟨*proof*⟩

lemma *domain-Diff-eq*: $[\![\langle a,c\rangle \in r;\ c{\neq}b]\!] \Longrightarrow domain(r{-}\{\langle a,b\rangle\}) = domain(r)$
⟨*proof*⟩

lemma *range-Diff-eq*: $[\![\langle c,b\rangle \in r;\ c{\neq}a]\!] \Longrightarrow range(r{-}\{\langle a,b\rangle\}) = range(r)$
⟨*proof*⟩

4.9 Image of a Set under a Function or Relation

lemma *image-iff*: $b \in r\text{``}A \longleftrightarrow (\exists\, x{\in}A.\ \langle x,b\rangle{\in}r)$
⟨*proof*⟩

lemma *image-singleton-iff*: $b \in r\text{``}\{a\} \longleftrightarrow \langle a,b\rangle{\in}r$

⟨*proof*⟩

lemma *imageI* [*intro*]: ⟦⟨*a*,*b*⟩∈ *r*; *a*∈*A*⟧ ⟹ *b* ∈ *r*‘‘*A*
⟨*proof*⟩

lemma *imageE* [*elim*!]:
  ⟦*b*: *r*‘‘*A*; ⋀*x*.⟦⟨*x*,*b*⟩∈ *r*; *x*∈*A*⟧ ⟹ *P*⟧ ⟹ *P*
⟨*proof*⟩

lemma *image-subset*: *r* ⊆ *A*∗*B* ⟹ *r*‘‘*C* ⊆ *B*
⟨*proof*⟩

lemma *image-0* [*simp*]: *r*‘‘*0* = *0*
⟨*proof*⟩

lemma *image-Un* [*simp*]: *r*‘‘(*A* ∪ *B*) = (*r*‘‘*A*) ∪ (*r*‘‘*B*)
⟨*proof*⟩

lemma *image-UN*: *r* ‘‘ (⋃*x*∈*A*. *B*(*x*)) = (⋃*x*∈*A*. *r* ‘‘ *B*(*x*))
⟨*proof*⟩

lemma *Collect-image-eq*:
  {*z* ∈ *Sigma*(*A*,*B*). *P*(*z*)} ‘‘ *C* = (⋃*x* ∈ *A*. {*y* ∈ *B*(*x*). *x* ∈ *C* ∧ *P*(⟨*x*,*y*⟩)})
⟨*proof*⟩

lemma *image-Int-subset*: *r*‘‘(*A* ∩ *B*) ⊆ (*r*‘‘*A*) ∩ (*r*‘‘*B*)
⟨*proof*⟩

lemma *image-Int-square-subset*: (*r* ∩ *A*∗*A*)‘‘*B* ⊆ (*r*‘‘*B*) ∩ *A*
⟨*proof*⟩

lemma *image-Int-square*: *B*⊆*A* ⟹ (*r* ∩ *A*∗*A*)‘‘*B* = (*r*‘‘*B*) ∩ *A*
⟨*proof*⟩

lemma *image-0-left* [*simp*]: *0*‘‘*A* = *0*
⟨*proof*⟩

lemma *image-Un-left*: (*r* ∪ *s*)‘‘*A* = (*r*‘‘*A*) ∪ (*s*‘‘*A*)
⟨*proof*⟩

lemma *image-Int-subset-left*: (*r* ∩ *s*)‘‘*A* ⊆ (*r*‘‘*A*) ∩ (*s*‘‘*A*)
⟨*proof*⟩

4.10 Inverse Image of a Set under a Function or Relation

lemma *vimage-iff*:
  a ∈ *r*−‘‘*B* ⟷ (∃ *y*∈*B*. ⟨*a*,*y*⟩∈*r*)

⟨*proof*⟩

lemma *vimage-singleton-iff*: $a \in r-``\{b\} \longleftrightarrow \langle a,b\rangle \in r$
⟨*proof*⟩

lemma *vimageI* [*intro*]: $[\![\langle a,b\rangle \in r;\ \ b \in B]\!] \Longrightarrow a \in r-``B$
⟨*proof*⟩

lemma *vimageE* [*elim*!]:
$[\![a: r-``B;\ \ \bigwedge x.[\![\langle a,x\rangle \in r;\ \ x \in B]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *vimage-subset*: $r \subseteq A*B \Longrightarrow r-``C \subseteq A$
⟨*proof*⟩

lemma *vimage-0* [*simp*]: $r-``0 = 0$
⟨*proof*⟩

lemma *vimage-Un* [*simp*]: $r-``(A \cup B) = (r-``A) \cup (r-``B)$
⟨*proof*⟩

lemma *vimage-Int-subset*: $r-``(A \cap B) \subseteq (r-``A) \cap (r-``B)$
⟨*proof*⟩

lemma *vimage-eq-UN*: $f\ -``B = (\bigcup y \in B.\ f-``\{y\})$
⟨*proof*⟩

lemma *function-vimage-Int*:
$function(f) \Longrightarrow f-``(A \cap B) = (f-``A)\ \cap\ (f-``B)$
⟨*proof*⟩

lemma *function-vimage-Diff*: $function(f) \Longrightarrow f-``(A-B) = (f-``A)\ -\ (f-``B)$
⟨*proof*⟩

lemma *function-image-vimage*: $function(f) \Longrightarrow f\ ``\ (f-``\ A) \subseteq A$
⟨*proof*⟩

lemma *vimage-Int-square-subset*: $(r \cap A*A)-``B \subseteq (r-``B) \cap A$
⟨*proof*⟩

lemma *vimage-Int-square*: $B \subseteq A \Longrightarrow (r \cap A*A)-``B = (r-``B) \cap A$
⟨*proof*⟩

lemma *vimage-0-left* [*simp*]: $0-``A = 0$
⟨*proof*⟩

lemma *vimage-Un-left*: $(r \cup s)-``A = (r-``A) \cup (s-``A)$
⟨*proof*⟩

lemma *vimage-Int-subset-left*: $(r \cap s)-``A \subseteq (r-``A) \cap (s-``A)$
⟨*proof*⟩

lemma *converse-Un* [*simp*]: $converse(A \cup B) = converse(A) \cup converse(B)$
⟨*proof*⟩

lemma *converse-Int* [*simp*]: $converse(A \cap B) = converse(A) \cap converse(B)$
⟨*proof*⟩

lemma *converse-Diff* [*simp*]: $converse(A - B) = converse(A) - converse(B)$
⟨*proof*⟩

lemma *converse-UN* [*simp*]: $converse(\bigcup x{\in}A.\ B(x)) = (\bigcup x{\in}A.\ converse(B(x)))$
⟨*proof*⟩

lemma *converse-INT* [*simp*]:
$converse(\bigcap x{\in}A.\ B(x)) = (\bigcap x{\in}A.\ converse(B(x)))$
⟨*proof*⟩

4.11 Powerset Operator

lemma *Pow-0* [*simp*]: $Pow(0) = \{0\}$
⟨*proof*⟩

lemma *Pow-insert*: $Pow\ (cons(a,A)) = Pow(A) \cup \{cons(a,X)\ .\ X{:}\ Pow(A)\}$
⟨*proof*⟩

lemma *Un-Pow-subset*: $Pow(A) \cup Pow(B) \subseteq Pow(A \cup B)$
⟨*proof*⟩

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow(B(x))) \subseteq Pow(\bigcup x{\in}A.\ B(x))$
⟨*proof*⟩

lemma *subset-Pow-Union*: $A \subseteq Pow(\bigcup(A))$
⟨*proof*⟩

lemma *Union-Pow-eq* [*simp*]: $\bigcup(Pow(A)) = A$
⟨*proof*⟩

lemma *Union-Pow-iff*: $\bigcup(A) \in Pow(B) \longleftrightarrow A \in Pow(Pow(B))$
⟨*proof*⟩

lemma *Pow-Int-eq* [*simp*]: $Pow(A \cap B) = Pow(A) \cap Pow(B)$
⟨*proof*⟩

lemma *Pow-INT-eq*: $A \neq 0 \Longrightarrow Pow(\bigcap x{\in}A.\ B(x)) = (\bigcap x{\in}A.\ Pow(B(x)))$
⟨*proof*⟩

4.12 RepFun

lemma *RepFun-subset*: $[\![\bigwedge x.\ x{\in}A \Longrightarrow f(x) \in B]\!] \Longrightarrow \{f(x).\ x{\in}A\} \subseteq B$
⟨*proof*⟩

lemma *RepFun-eq-0-iff* [*simp*]: $\{f(x).x{\in}A\}=0 \longleftrightarrow A=0$
⟨*proof*⟩

lemma *RepFun-constant* [*simp*]: $\{c.\ x{\in}A\} = (\mathit{if}\ A=0\ \mathit{then}\ 0\ \mathit{else}\ \{c\})$
⟨*proof*⟩

4.13 Collect

lemma *Collect-subset*: $Collect(A,P) \subseteq A$
⟨*proof*⟩

lemma *Collect-Un*: $Collect(A \cup B,\ P) = Collect(A,P) \cup Collect(B,P)$
⟨*proof*⟩

lemma *Collect-Int*: $Collect(A \cap B,\ P) = Collect(A,P) \cap Collect(B,P)$
⟨*proof*⟩

lemma *Collect-Diff*: $Collect(A - B,\ P) = Collect(A,P) - Collect(B,P)$
⟨*proof*⟩

lemma *Collect-cons*: $\{x{\in}cons(a,B).\ P(x)\} =$
$(\mathit{if}\ P(a)\ \mathit{then}\ cons(a,\ \{x{\in}B.\ P(x)\})\ \mathit{else}\ \{x{\in}B.\ P(x)\})$
⟨*proof*⟩

lemma *Int-Collect-self-eq*: $A \cap Collect(A,P) = Collect(A,P)$
⟨*proof*⟩

lemma *Collect-Collect-eq* [*simp*]:
$Collect(Collect(A,P),\ Q) = Collect(A,\ \lambda x.\ P(x) \wedge Q(x))$
⟨*proof*⟩

lemma *Collect-Int-Collect-eq*:
$Collect(A,P) \cap Collect(A,Q) = Collect(A,\ \lambda x.\ P(x) \wedge Q(x))$
⟨*proof*⟩

lemma *Collect-Union-eq* [*simp*]:
$Collect(\bigcup x{\in}A.\ B(x),\ P) = (\bigcup x{\in}A.\ Collect(B(x),\ P))$
⟨*proof*⟩

lemma *Collect-Int-left*: $\{x{\in}A.\ P(x)\} \cap B = \{x \in A \cap B.\ P(x)\}$
⟨*proof*⟩

lemma *Collect-Int-right*: $A \cap \{x{\in}B.\ P(x)\} = \{x \in A \cap B.\ P(x)\}$
⟨*proof*⟩

lemma *Collect-disj-eq*: $\{x{\in}A.\ P(x) \mid Q(x)\} = Collect(A,\ P) \cup Collect(A,\ Q)$
⟨*proof*⟩

lemma *Collect-conj-eq*: $\{x{\in}A.\ P(x) \wedge Q(x)\} = Collect(A,\ P) \cap Collect(A,\ Q)$
⟨*proof*⟩

lemmas *subset-SIs = subset-refl cons-subsetI subset-consI*
Union-least UN-least Un-least
Inter-greatest Int-greatest RepFun-subset
Un-upper1 Un-upper2 Int-lower1 Int-lower2

⟨*ML*⟩

end

5 Least and Greatest Fixed Points; the Knaster-Tarski Theorem

theory *Fixedpt* **imports** *equalities* **begin**

definition

bnd-mono :: $[i,i{\Rightarrow}i]{\Rightarrow}o$ **where**
bnd-mono$(D,h) \equiv h(D){<=}D \wedge (\forall\, W\ X.\ W{<=}X \longrightarrow X{<=}D \longrightarrow h(W) \subseteq h(X))$

definition
lfp :: $[i,i{\Rightarrow}i]{\Rightarrow}i$ **where**
lfp$(D,h) \equiv \bigcap(\{X{\colon}\ Pow(D).\ h(X) \subseteq X\})$

definition
gfp :: $[i,i{\Rightarrow}i]{\Rightarrow}i$ **where**
gfp$(D,h) \equiv \bigcup(\{X{\colon}\ Pow(D).\ X \subseteq h(X)\})$

The theorem is proved in the lattice of subsets of D, namely $Pow(D)$, with Inter as the greatest lower bound.

5.1 Monotone Operators

lemma *bnd-monoI*:
$[\![h(D){<=}D;$

$\bigwedge W\ X.\ [\![W<=D;\ \ X<=D;\ \ W<=X]\!] \Longrightarrow h(W) \subseteq h(X)$
$]\!] \Longrightarrow$ *bnd-mono*(*D*,*h*)
⟨*proof*⟩

lemma *bnd-monoD1*: *bnd-mono*(*D*,*h*) $\Longrightarrow h(D) \subseteq D$
⟨*proof*⟩

lemma *bnd-monoD2*: $[\![$*bnd-mono*(*D*,*h*); $\ W<=X;\ \ X<=D]\!] \Longrightarrow h(W) \subseteq h(X)$
⟨*proof*⟩

lemma *bnd-mono-subset*:
$[\![$*bnd-mono*(*D*,*h*); $\ X<=D]\!] \Longrightarrow h(X) \subseteq D$
⟨*proof*⟩

lemma *bnd-mono-Un*:
$[\![$*bnd-mono*(*D*,*h*); $\ A \subseteq D;\ \ B \subseteq D]\!] \Longrightarrow h(A) \cup h(B) \subseteq h(A \cup B)$
⟨*proof*⟩

lemma *bnd-mono-UN*:
$[\![$*bnd-mono*(*D*,*h*); $\ \forall i{\in}I.\ A(i) \subseteq D]\!]$
$\Longrightarrow (\bigcup i{\in}I.\ h(A(i))) \subseteq h((\bigcup i{\in}I.\ A(i)))$
⟨*proof*⟩

lemma *bnd-mono-Int*:
$[\![$*bnd-mono*(*D*,*h*); $\ A \subseteq D;\ \ B \subseteq D]\!] \Longrightarrow h(A \cap B) \subseteq h(A) \cap h(B)$
⟨*proof*⟩

5.2 Proof of Knaster-Tarski Theorem using *lfp*

lemma *lfp-lowerbound*:
$[\![h(A) \subseteq A;\ \ A<=D]\!] \Longrightarrow$ *lfp*(*D*,*h*) $\subseteq A$
⟨*proof*⟩

lemma *lfp-subset*: *lfp*(*D*,*h*) $\subseteq D$
⟨*proof*⟩

lemma *def-lfp-subset*: $\ A \equiv$ *lfp*(*D*,*h*) $\Longrightarrow A \subseteq D$
⟨*proof*⟩

lemma *lfp-greatest*:
$[\![h(D) \subseteq D;\ \bigwedge X.\ [\![h(X) \subseteq X;\ \ X<=D]\!] \Longrightarrow A<=X]\!] \Longrightarrow A \subseteq$ *lfp*(*D*,*h*)
⟨*proof*⟩

lemma *lfp-lemma1*:
$[\![$*bnd-mono*(*D*,*h*); $\ h(A)<=A;\ \ A<=D]\!] \Longrightarrow h($*lfp*(*D*,*h*)$) \subseteq A$

⟨*proof*⟩

lemma *lfp-lemma2*: *bnd-mono*(*D*,*h*) ⟹ *h*(*lfp*(*D*,*h*)) ⊆ *lfp*(*D*,*h*)
⟨*proof*⟩

lemma *lfp-lemma3*:
bnd-mono(*D*,*h*) ⟹ *lfp*(*D*,*h*) ⊆ *h*(*lfp*(*D*,*h*))
⟨*proof*⟩

lemma *lfp-unfold*: *bnd-mono*(*D*,*h*) ⟹ *lfp*(*D*,*h*) = *h*(*lfp*(*D*,*h*))
⟨*proof*⟩

lemma *def-lfp-unfold*:
⟦*A*≡*lfp*(*D*,*h*); *bnd-mono*(*D*,*h*)⟧ ⟹ *A* = *h*(*A*)
⟨*proof*⟩

5.3 General Induction Rule for Least Fixedpoints

lemma *Collect-is-pre-fixedpt*:
⟦*bnd-mono*(*D*,*h*); ⋀*x*. *x* ∈ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⟹ *P*(*x*)⟧
⟹ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⊆ *Collect*(*lfp*(*D*,*h*),*P*)
⟨*proof*⟩

lemma *induct*:
⟦*bnd-mono*(*D*,*h*); *a* ∈ *lfp*(*D*,*h*);
⋀*x*. *x* ∈ *h*(*Collect*(*lfp*(*D*,*h*),*P*)) ⟹ *P*(*x*)
⟧ ⟹ *P*(*a*)
⟨*proof*⟩

lemma *def-induct*:
⟦*A* ≡ *lfp*(*D*,*h*); *bnd-mono*(*D*,*h*); *a*:*A*;
⋀*x*. *x* ∈ *h*(*Collect*(*A*,*P*)) ⟹ *P*(*x*)
⟧ ⟹ *P*(*a*)
⟨*proof*⟩

lemma *lfp-Int-lowerbound*:
⟦*h*(*D* ∩ *A*) ⊆ *A*; *bnd-mono*(*D*,*h*)⟧ ⟹ *lfp*(*D*,*h*) ⊆ *A*
⟨*proof*⟩

lemma *lfp-mono*:
assumes *hmono*: *bnd-mono*(*D*,*h*)
and *imono*: *bnd-mono*(*E*,*i*)
and *subhi*: ⋀*X*. *X*<=*D* ⟹ *h*(*X*) ⊆ *i*(*X*)
shows *lfp*(*D*,*h*) ⊆ *lfp*(*E*,*i*)

⟨*proof*⟩

lemma *lfp-mono2*:
⟦$i(D) \subseteq D$; $\bigwedge X.\ X<=D \Longrightarrow h(X) \subseteq i(X)$⟧ $\Longrightarrow$ *lfp*(D,h) $\subseteq$ *lfp*(D,i)
⟨*proof*⟩

lemma *lfp-cong*:
⟦$D=D'$; $\bigwedge X.\ X \subseteq D' \Longrightarrow h(X) = h'(X)$⟧ $\Longrightarrow$ *lfp*(D,h) = *lfp*(D',h')
⟨*proof*⟩

5.4 Proof of Knaster-Tarski Theorem using *gfp*

lemma *gfp-upperbound*: ⟦$A \subseteq h(A)$; $A<=D$⟧ $\Longrightarrow$ $A \subseteq$ *gfp*(D,h)
⟨*proof*⟩

lemma *gfp-subset*: *gfp*(D,h) $\subseteq$ D
⟨*proof*⟩

lemma *def-gfp-subset*: A≡*gfp*(D,h) $\Longrightarrow$ $A \subseteq D$
⟨*proof*⟩

lemma *gfp-least*:
⟦*bnd-mono*(D,h); $\bigwedge X.$ ⟦$X \subseteq h(X)$; $X<=D$⟧ $\Longrightarrow$ $X<=A$⟧ $\Longrightarrow$
gfp(D,h) $\subseteq$ A
⟨*proof*⟩

lemma *gfp-lemma1*:
⟦*bnd-mono*(D,h); $A<=h(A)$; $A<=D$⟧ $\Longrightarrow$ $A \subseteq h($*gfp*(D,h)$)$
⟨*proof*⟩

lemma *gfp-lemma2*: *bnd-mono*(D,h) $\Longrightarrow$ *gfp*(D,h) $\subseteq$ $h($*gfp*(D,h)$)$
⟨*proof*⟩

lemma *gfp-lemma3*:
bnd-mono(D,h) $\Longrightarrow$ $h($*gfp*(D,h)$) \subseteq$ *gfp*(D,h)
⟨*proof*⟩

lemma *gfp-unfold*: *bnd-mono*(D,h) $\Longrightarrow$ *gfp*(D,h) = $h($*gfp*(D,h)$)$
⟨*proof*⟩

lemma *def-gfp-unfold*:
⟦A≡*gfp*(D,h); *bnd-mono*(D,h)⟧ $\Longrightarrow$ $A = h(A)$
⟨*proof*⟩

5.5 Coinduction Rules for Greatest Fixed Points

lemma *weak-coinduct*: ⟦a: X; $X \subseteq h(X)$; $X \subseteq D$⟧ $\Longrightarrow$ $a \in$ *gfp*(D,h)

⟨*proof*⟩

lemma *coinduct-lemma*:
⟦$X \subseteq h(X \cup gfp(D,h))$; $X \subseteq D$; *bnd-mono*(D,h)⟧ ⟹
$X \cup gfp(D,h) \subseteq h(X \cup gfp(D,h))$
⟨*proof*⟩

lemma *coinduct*:
⟦*bnd-mono*(D,h); a: X; $X \subseteq h(X \cup gfp(D,h))$; $X \subseteq D$⟧
⟹ $a \in gfp(D,h)$
⟨*proof*⟩

lemma *def-coinduct*:
⟦$A \equiv gfp(D,h)$; *bnd-mono*(D,h); a: X; $X \subseteq h(X \cup A)$; $X \subseteq D$⟧ ⟹
$a \in A$
⟨*proof*⟩

lemma *def-Collect-coinduct*:
⟦$A \equiv gfp(D, \lambda w.\ Collect(D,P(w)))$; *bnd-mono*$(D, \lambda w.\ Collect(D,P(w)))$;
a: X; $X \subseteq D$; $\bigwedge z.\ z$: $X \Longrightarrow P(X \cup A, z)$⟧ ⟹
$a \in A$
⟨*proof*⟩

lemma *gfp-mono*:
⟦*bnd-mono*(D,h); $D \subseteq E$;
$\bigwedge X.\ X <= D \Longrightarrow h(X) \subseteq i(X)$⟧ ⟹ $gfp(D,h) \subseteq gfp(E,i)$
⟨*proof*⟩

end

6 Booleans in Zermelo-Fraenkel Set Theory

theory *Bool* **imports** *pair* **begin**

abbreviation
one (‹*1*›) **where**
1 ≡ *succ*(*0*)

abbreviation
two (‹*2*›) **where**
2 ≡ *succ*(*1*)

2 is equal to bool, but is used as a number rather than a type.

definition *bool* ≡ {*0*,*1*}

definition $cond(b,c,d) \equiv if(b=1,c,d)$

definition $not(b) \equiv cond(b,0,1)$

definition
  and :: $[i,i]\Rightarrow i$ (**infixl** ‹and› 70) **where**
    $a\ and\ b \equiv cond(a,b,0)$

definition
  or :: $[i,i]\Rightarrow i$ (**infixl** ‹or› 65) **where**
    $a\ or\ b \equiv cond(a,1,b)$

definition
  xor :: $[i,i]\Rightarrow i$ (**infixl** ‹xor› 65) **where**
    $a\ xor\ b \equiv cond(a,not(b),b)$

lemmas *bool-defs* = *bool-def cond-def*

lemma *singleton-0*: $\{0\} = 1$
⟨*proof*⟩

lemma *bool-1I* [*simp*, *TC*]: $1 \in bool$
⟨*proof*⟩

lemma *bool-0I* [*simp*, *TC*]: $0 \in bool$
⟨*proof*⟩

lemma *one-not-0*: $1 \neq 0$
⟨*proof*⟩

lemmas *one-neq-0* = *one-not-0* [*THEN notE*]

lemma *boolE*:
  $[\![c: bool;\ c=1 \Longrightarrow P;\ c=0 \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *cond-1* [*simp*]: $cond(1,c,d) = c$
⟨*proof*⟩

lemma *cond-0* [*simp*]: $cond(0,c,d) = d$
⟨*proof*⟩

lemma *cond-type* [*TC*]: ⟦*b*: *bool*; *c*: *A*(*1*); *d*: *A*(*0*)⟧ ⟹ *cond*(*b*,*c*,*d*): *A*(*b*)
⟨*proof*⟩

lemma *cond-simple-type*: ⟦*b*: *bool*; *c*: *A*; *d*: *A*⟧ ⟹ *cond*(*b*,*c*,*d*): *A*
⟨*proof*⟩

lemma *def-cond-1*: ⟦⋀*b*. *j*(*b*)≡*cond*(*b*,*c*,*d*)⟧ ⟹ *j*(*1*) = *c*
⟨*proof*⟩

lemma *def-cond-0*: ⟦⋀*b*. *j*(*b*)≡*cond*(*b*,*c*,*d*)⟧ ⟹ *j*(*0*) = *d*
⟨*proof*⟩

lemmas *not-1* = *not-def* [*THEN def-cond-1*, *simp*]
lemmas *not-0* = *not-def* [*THEN def-cond-0*, *simp*]

lemmas *and-1* = *and-def* [*THEN def-cond-1*, *simp*]
lemmas *and-0* = *and-def* [*THEN def-cond-0*, *simp*]

lemmas *or-1* = *or-def* [*THEN def-cond-1*, *simp*]
lemmas *or-0* = *or-def* [*THEN def-cond-0*, *simp*]

lemmas *xor-1* = *xor-def* [*THEN def-cond-1*, *simp*]
lemmas *xor-0* = *xor-def* [*THEN def-cond-0*, *simp*]

lemma *not-type* [*TC*]: *a*:*bool* ⟹ *not*(*a*) ∈ *bool*
⟨*proof*⟩

lemma *and-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a and b* ∈ *bool*
⟨*proof*⟩

lemma *or-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a or b* ∈ *bool*
⟨*proof*⟩

lemma *xor-type* [*TC*]: ⟦*a*:*bool*; *b*:*bool*⟧ ⟹ *a xor b* ∈ *bool*
⟨*proof*⟩

lemmas *bool-typechecks* = *bool-1I bool-0I cond-type not-type and-type*
or-type xor-type

6.1 Laws About 'not'

lemma *not-not* [*simp*]: *a*:*bool* ⟹ *not*(*not*(*a*)) = *a*
⟨*proof*⟩

lemma *not-and* [*simp*]: *a*:*bool* ⟹ *not*(*a and b*) = *not*(*a*) *or not*(*b*)
⟨*proof*⟩

lemma *not-or* [*simp*]: *a*:*bool* $\Longrightarrow$ *not*(*a or b*) = *not*(*a*) *and not*(*b*)
⟨*proof*⟩

6.2 Laws About 'and'

lemma *and-absorb* [*simp*]: *a*: *bool* $\Longrightarrow$ *a and a* = *a*
⟨*proof*⟩

lemma *and-commute*: ⟦*a*: *bool*; *b*:*bool*⟧ $\Longrightarrow$ *a and b* = *b and a*
⟨*proof*⟩

lemma *and-assoc*: *a*: *bool* $\Longrightarrow$ (*a and b*) *and c* = *a and* (*b and c*)
⟨*proof*⟩

lemma *and-or-distrib*: ⟦*a*: *bool*; *b*:*bool*; *c*:*bool*⟧ $\Longrightarrow$
(*a or b*) *and c* = (*a and c*) *or* (*b and c*)
⟨*proof*⟩

6.3 Laws About 'or'

lemma *or-absorb* [*simp*]: *a*: *bool* $\Longrightarrow$ *a or a* = *a*
⟨*proof*⟩

lemma *or-commute*: ⟦*a*: *bool*; *b*:*bool*⟧ $\Longrightarrow$ *a or b* = *b or a*
⟨*proof*⟩

lemma *or-assoc*: *a*: *bool* $\Longrightarrow$ (*a or b*) *or c* = *a or* (*b or c*)
⟨*proof*⟩

lemma *or-and-distrib*: ⟦*a*: *bool*; *b*: *bool*; *c*: *bool*⟧ $\Longrightarrow$
(*a and b*) *or c* = (*a or c*) *and* (*b or c*)
⟨*proof*⟩

definition
bool-of-o :: *o*⇒*i* **where**
bool-of-o(*P*) ≡ (*if P then 1 else 0*)

lemma [*simp*]: *bool-of-o*(*True*) = *1*
⟨*proof*⟩

lemma [*simp*]: *bool-of-o*(*False*) = *0*
⟨*proof*⟩

lemma [*simp*,*TC*]: *bool-of-o*(*P*) ∈ *bool*
⟨*proof*⟩

lemma [*simp*]: (*bool-of-o*(*P*) = *1*) $\longleftrightarrow$ *P*
⟨*proof*⟩

lemma [*simp*]: (*bool-of-o*(*P*) = *0*) ⟷ ¬*P*
⟨*proof*⟩

end

7 Disjoint Sums

theory *Sum* **imports** *Bool equalities* **begin**

And the "Part" primitive for simultaneous recursive type definitions

definition *sum* :: [*i*,*i*]⇒*i* (**infixr** ‹+› *65*) **where**
    A+*B* ≡ {*0*}∗*A* ∪ {*1*}∗*B*

definition *Inl* :: *i*⇒*i* **where**
    Inl(*a*) ≡ ⟨*0*,*a*⟩

definition *Inr* :: *i*⇒*i* **where**
    Inr(*b*) ≡ ⟨*1*,*b*⟩

definition *case* :: [*i*⇒*i*, *i*⇒*i*, *i*]⇒*i* **where**
    case(*c*,*d*) ≡ (λ⟨*y*,*z*⟩. *cond*(*y*, *d*(*z*), *c*(*z*)))

definition *Part* :: [*i*,*i*⇒*i*] ⇒ *i* **where**
    Part(*A*,*h*) ≡ {*x* ∈ *A*. ∃ *z*. *x* = *h*(*z*)}

7.1 Rules for the *Part* Primitive

lemma *Part-iff*:
   a ∈ *Part*(*A*,*h*) ⟷ *a* ∈ *A* ∧ (∃ *y*. *a*=*h*(*y*))
  ⟨*proof*⟩

lemma *Part-eqI* [*intro*]:
   ⟦*a* ∈ *A*; *a*=*h*(*b*)⟧ ⟹ *a* ∈ *Part*(*A*,*h*)
⟨*proof*⟩

lemmas *PartI* = *refl* [*THEN* [*2*] *Part-eqI*]

lemma *PartE* [*elim*!]:
   ⟦*a* ∈ *Part*(*A*,*h*); ⋀*z*. ⟦*a* ∈ *A*; *a*=*h*(*z*)⟧ ⟹ *P*
⟧ ⟹ *P*
⟨*proof*⟩

lemma *Part-subset*: *Part*(*A*,*h*) ⊆ *A*
  ⟨*proof*⟩

7.2 Rules for Disjoint Sums

lemmas *sum-defs* = *sum-def Inl-def Inr-def case-def*

lemma *Sigma-bool*: $Sigma(bool,C) = C(0) + C(1)$
⟨*proof*⟩

lemma *InlI* [*intro*!,*simp*,*TC*]: $a \in A \Longrightarrow Inl(a) \in A+B$
⟨*proof*⟩

lemma *InrI* [*intro*!,*simp*,*TC*]: $b \in B \Longrightarrow Inr(b) \in A+B$
⟨*proof*⟩

lemma *sumE* [*elim*!]:
$[\![u \in A+B;$
$\bigwedge x.\ [\![x \in A;\ \ u=Inl(x)]\!] \Longrightarrow P;$
$\bigwedge y.\ [\![y \in B;\ \ u=Inr(y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *Inl-iff* [*iff*]: $Inl(a)=Inl(b) \longleftrightarrow a=b$
⟨*proof*⟩

lemma *Inr-iff* [*iff*]: $Inr(a)=Inr(b) \longleftrightarrow a=b$
⟨*proof*⟩

lemma *Inl-Inr-iff* [*simp*]: $Inl(a)=Inr(b) \longleftrightarrow False$
⟨*proof*⟩

lemma *Inr-Inl-iff* [*simp*]: $Inr(b)=Inl(a) \longleftrightarrow False$
⟨*proof*⟩

lemma *sum-empty* [*simp*]: $0+0 = 0$
⟨*proof*⟩

lemmas *Inl-inject* = *Inl-iff* [*THEN iffD1*]
lemmas *Inr-inject* = *Inr-iff* [*THEN iffD1*]
lemmas *Inl-neq-Inr* = *Inl-Inr-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]
lemmas *Inr-neq-Inl* = *Inr-Inl-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]

lemma *InlD*: $Inl(a)$: $A+B \Longrightarrow a \in A$
⟨*proof*⟩

lemma *InrD*: $Inr(b)$: $A+B \Longrightarrow b \in B$
⟨*proof*⟩

lemma *sum-iff*: $u \in A+B \longleftrightarrow (\exists\, x.\ x \in A \wedge u{=}Inl(x)) \mid (\exists\, y.\ y \in B \wedge u{=}Inr(y))$
⟨*proof*⟩

lemma *Inl-in-sum-iff* [*simp*]: $(Inl(x) \in A+B) \longleftrightarrow (x \in A)$
⟨*proof*⟩

lemma *Inr-in-sum-iff* [*simp*]: $(Inr(y) \in A+B) \longleftrightarrow (y \in B)$
⟨*proof*⟩

lemma *sum-subset-iff*: $A+B \subseteq C+D \longleftrightarrow A{<=}C \wedge B{<=}D$
⟨*proof*⟩

lemma *sum-equal-iff*: $A+B = C+D \longleftrightarrow A{=}C \wedge B{=}D$
⟨*proof*⟩

lemma *sum-eq-2-times*: $A+A = 2{*}A$
⟨*proof*⟩

7.3 The Eliminator: *case*

lemma *case-Inl* [*simp*]: $case(c,\ d,\ Inl(a)) = c(a)$
⟨*proof*⟩

lemma *case-Inr* [*simp*]: $case(c,\ d,\ Inr(b)) = d(b)$
⟨*proof*⟩

lemma *case-type* [*TC*]:
$[\![u \in A+B;$
$\bigwedge x.\ x \in A \Longrightarrow c(x)\colon C(Inl(x));$
$\bigwedge y.\ y \in B \Longrightarrow d(y)\colon C(Inr(y))$
$]\!] \Longrightarrow case(c,d,u) \in C(u)$
⟨*proof*⟩

lemma *expand-case*: $u \in A+B \Longrightarrow$
$R(case(c,d,u)) \longleftrightarrow$
$((\forall\, x{\in}A.\ u = Inl(x) \longrightarrow R(c(x))) \wedge$
$(\forall\, y{\in}B.\ u = Inr(y) \longrightarrow R(d(y))))$
⟨*proof*⟩

lemma *case-cong*:
$[\![z \in A+B;$
$\bigwedge x.\ x \in A \Longrightarrow c(x){=}c'(x);$
$\bigwedge y.\ y \in B \Longrightarrow d(y){=}d'(y)$
$]\!] \Longrightarrow case(c,d,z) = case(c',d',z)$
⟨*proof*⟩

lemma *case-case*: $z \in A+B \Longrightarrow$
$case(c,\ d,\ case(\lambda x.\ Inl(c'(x)),\ \lambda y.\ Inr(d'(y)),\ z)) =$
$case(\lambda x.\ c(c'(x)),\ \lambda y.\ d(d'(y)),\ z)$
⟨*proof*⟩

7.4 More Rules for $Part(A,\ h)$

lemma *Part-mono*: $A<=B \Longrightarrow Part(A,h)<=Part(B,h)$
⟨*proof*⟩

lemma *Part-Collect*: $Part(Collect(A,P),\ h) = Collect(Part(A,h),\ P)$
⟨*proof*⟩

lemmas *Part-CollectE* =
Part-Collect [*THEN equalityD1*, *THEN subsetD*, *THEN CollectE*]

lemma *Part-Inl*: $Part(A+B,Inl) = \{Inl(x).\ x \in A\}$
⟨*proof*⟩

lemma *Part-Inr*: $Part(A+B,Inr) = \{Inr(y).\ y \in B\}$
⟨*proof*⟩

lemma *PartD1*: $a \in Part(A,h) \Longrightarrow a \in A$
⟨*proof*⟩

lemma *Part-id*: $Part(A,\lambda x.\ x) = A$
⟨*proof*⟩

lemma *Part-Inr2*: $Part(A+B,\ \lambda x.\ Inr(h(x))) = \{Inr(y).\ y \in Part(B,h)\}$
⟨*proof*⟩

lemma *Part-sum-equality*: $C \subseteq A+B \Longrightarrow Part(C,Inl) \cup Part(C,Inr) = C$
⟨*proof*⟩

end

8 Functions, Function Spaces, Lambda-Abstraction

theory *func* **imports** *equalities Sum* **begin**

8.1 The Pi Operator: Dependent Function Space

lemma *subset-Sigma-imp-relation*: $r \subseteq Sigma(A,B) \Longrightarrow relation(r)$
⟨*proof*⟩

lemma *relation-converse-converse* [*simp*]:
$relation(r) \Longrightarrow converse(converse(r)) = r$
⟨*proof*⟩

lemma *relation-restrict* [*simp*]: *relation*(*restrict*(*r*,*A*))
⟨*proof*⟩

lemma *Pi-iff*:
f ∈ *Pi*(*A*,*B*) ⟷ *function*(*f*) ∧ *f*<=*Sigma*(*A*,*B*) ∧ *A*<=*domain*(*f*)
⟨*proof*⟩

lemma *Pi-iff-old*:
f ∈ *Pi*(*A*,*B*) ⟷ *f*<=*Sigma*(*A*,*B*) ∧ (∀ *x*∈*A*. ∃!*y*. ⟨*x*,*y*⟩: *f*)
⟨*proof*⟩

lemma *fun-is-function*: *f* ∈ *Pi*(*A*,*B*) ⟹ *function*(*f*)
⟨*proof*⟩

lemma *function-imp-Pi*:
⟦*function*(*f*); *relation*(*f*)⟧ ⟹ *f* ∈ *domain*(*f*) -> *range*(*f*)
⟨*proof*⟩

lemma *functionI*:
⟦⋀*x y y'*. ⟦⟨*x*,*y*⟩:*r*; <*x*,*y'*>:*r*⟧ ⟹ *y*=*y'*⟧ ⟹ *function*(*r*)
⟨*proof*⟩

lemma *fun-is-rel*: *f* ∈ *Pi*(*A*,*B*) ⟹ *f* ⊆ *Sigma*(*A*,*B*)
⟨*proof*⟩

lemma *Pi-cong*:
⟦*A*=*A'*; ⋀*x*. *x* ∈ *A'* ⟹ *B*(*x*)=*B'*(*x*)⟧ ⟹ *Pi*(*A*,*B*) = *Pi*(*A'*,*B'*)
⟨*proof*⟩

lemma *fun-weaken-type*: ⟦*f* ∈ *A*->*B*; *B*<=*D*⟧ ⟹ *f* ∈ *A*->*D*
⟨*proof*⟩

8.2 Function Application

lemma *apply-equality2*: ⟦⟨*a*,*b*⟩: *f*; ⟨*a*,*c*⟩: *f*; *f* ∈ *Pi*(*A*,*B*)⟧ ⟹ *b*=*c*
⟨*proof*⟩

lemma *function-apply-equality*: ⟦⟨*a*,*b*⟩: *f*; *function*(*f*)⟧ ⟹ *f*‘*a* = *b*
⟨*proof*⟩

lemma *apply-equality*: ⟦⟨*a*,*b*⟩: *f*; *f* ∈ *Pi*(*A*,*B*)⟧ ⟹ *f*‘*a* = *b*
⟨*proof*⟩

lemma *apply-0*: $a \notin domain(f) \Longrightarrow f`a = 0$
⟨*proof*⟩

lemma *Pi-memberD*: $[\![f \in Pi(A,B);\ \ c \in f]\!] \Longrightarrow \exists x{\in}A.\ \ c = <x,f`x>$
⟨*proof*⟩

lemma *function-apply-Pair*: $[\![function(f);\ \ a \in domain(f)]\!] \Longrightarrow <a,f`a> : f$
⟨*proof*⟩

lemma *apply-Pair*: $[\![f \in Pi(A,B);\ \ a \in A]\!] \Longrightarrow <a,f`a> : f$
⟨*proof*⟩

lemma *apply-type* [*TC*]: $[\![f \in Pi(A,B);\ \ a \in A]\!] \Longrightarrow f`a \in B(a)$
⟨*proof*⟩

lemma *apply-funtype*: $[\![f \in A{->}B;\ \ a \in A]\!] \Longrightarrow f`a \in B$
⟨*proof*⟩

lemma *apply-iff*: $f \in Pi(A,B) \Longrightarrow \langle a,b\rangle : f \longleftrightarrow a \in A \wedge f`a = b$
⟨*proof*⟩

lemma *Pi-type*: $[\![f \in Pi(A,C);\ \ \bigwedge x.\ x \in A \Longrightarrow f`x \in B(x)]\!] \Longrightarrow f \in Pi(A,B)$
⟨*proof*⟩

lemma *Pi-Collect-iff*:
$(f \in Pi(A,\ \lambda x.\ \{y \in B(x).\ P(x,y)\}))$
$\longleftrightarrow\ f \in Pi(A,B) \wedge (\forall x{\in}A.\ P(x,\ f`x))$
⟨*proof*⟩

lemma *Pi-weaken-type*:
$[\![f \in Pi(A,B);\ \ \bigwedge x.\ x \in A \Longrightarrow B(x){<=}C(x)]\!] \Longrightarrow f \in Pi(A,C)$
⟨*proof*⟩

lemma *domain-type*: $[\![\langle a,b\rangle \in f;\ \ f \in Pi(A,B)]\!] \Longrightarrow a \in A$
⟨*proof*⟩

lemma *range-type*: $[\![\langle a,b\rangle \in f;\ \ f \in Pi(A,B)]\!] \Longrightarrow b \in B(a)$
⟨*proof*⟩

lemma *Pair-mem-PiD*: $[\![\langle a,b\rangle : f;\ \ f \in Pi(A,B)]\!] \Longrightarrow a \in A \wedge b \in B(a) \wedge f`a = b$
⟨*proof*⟩

8.3 Lambda Abstraction

lemma *lamI*: $a \in A \Longrightarrow <a,b(a)> \in (\lambda x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *lamE*:
$[\![p{:}\ (\lambda x{\in}A.\ b(x));\ \bigwedge x.[\![x \in A;\ p{=}<x,b(x)>]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *lamD*: $[\![\langle a,c\rangle{:}\ (\lambda x{\in}A.\ b(x))]\!] \Longrightarrow c = b(a)$
⟨*proof*⟩

lemma *lam-type* [*TC*]:
$[\![\bigwedge x.\ x \in A \Longrightarrow b(x){:}\ B(x)]\!] \Longrightarrow (\lambda x{\in}A.\ b(x)) \in Pi(A,B)$
⟨*proof*⟩

lemma *lam-funtype*: $(\lambda x{\in}A.\ b(x)) \in A \mathrel{->} \{b(x).\ x \in A\}$
⟨*proof*⟩

lemma *function-lam*: *function* $(\lambda x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *relation-lam*: *relation* $(\lambda x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *beta-if* [*simp*]: $(\lambda x{\in}A.\ b(x))\ `\ a = (\mathit{if}\ a \in A\ \mathit{then}\ b(a)\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *beta*: $a \in A \Longrightarrow (\lambda x{\in}A.\ b(x))\ `\ a = b(a)$
⟨*proof*⟩

lemma *lam-empty* [*simp*]: $(\lambda x{\in}0.\ b(x)) = 0$
⟨*proof*⟩

lemma *domain-lam* [*simp*]: $domain(Lambda(A,b)) = A$
⟨*proof*⟩

lemma *lam-cong* [*cong*]:
$[\![A{=}A';\ \bigwedge x.\ x \in A' \Longrightarrow b(x){=}b'(x)]\!] \Longrightarrow Lambda(A,b) = Lambda(A',b')$
⟨*proof*⟩

lemma *lam-theI*:
$(\bigwedge x.\ x \in A \Longrightarrow \exists!y.\ Q(x,y)) \Longrightarrow \exists f.\ \forall x{\in}A.\ Q(x,\ f`x)$
⟨*proof*⟩

lemma *lam-eqE*: $[\![(\lambda x{\in}A.\ f(x)) = (\lambda x{\in}A.\ g(x));\ \ a \in A]\!] \Longrightarrow f(a){=}g(a)$
⟨*proof*⟩

lemma *Pi-empty1* [*simp*]: $Pi(0,A) = \{0\}$
⟨*proof*⟩

lemma *singleton-fun* [*simp*]: $\{\langle a,b\rangle\} \in \{a\} -> \{b\}$
⟨*proof*⟩

lemma *Pi-empty2* [*simp*]: $(A->0) = (\mathit{if}\ A=0\ \mathit{then}\ \{0\}\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *fun-space-empty-iff* [*iff*]: $(A->X)=0 \longleftrightarrow X=0 \wedge (A \neq 0)$
⟨*proof*⟩

8.4 Extensionality

lemma *fun-subset*:
$[\![f \in Pi(A,B);\ g \in Pi(C,D);\ A<=C;$
$\bigwedge x.\ x \in A \Longrightarrow f`x = g`x]\!] \Longrightarrow f<=g$
⟨*proof*⟩

lemma *fun-extension*:
$[\![f \in Pi(A,B);\ g \in Pi(A,D);$
$\bigwedge x.\ x \in A \Longrightarrow f`x = g`x]\!] \Longrightarrow f=g$
⟨*proof*⟩

lemma *eta* [*simp*]: $f \in Pi(A,B) \Longrightarrow (\lambda x{\in}A.\ f`x) = f$
⟨*proof*⟩

lemma *fun-extension-iff*:
$[\![f \in Pi(A,B);\ g \in Pi(A,C)]\!] \Longrightarrow (\forall a{\in}A.\ f`a = g`a) \longleftrightarrow f=g$
⟨*proof*⟩

lemma *fun-subset-eq*: $[\![f \in Pi(A,B);\ g \in Pi(A,C)]\!] \Longrightarrow f \subseteq g \longleftrightarrow (f = g)$
⟨*proof*⟩

lemma *Pi-lamE*:
assumes *major*: $f \in Pi(A,B)$
and *minor*: $\bigwedge b.\ [\![\forall x{\in}A.\ b(x){:}B(x);\ f = (\lambda x{\in}A.\ b(x))]\!] \Longrightarrow P$
shows P
⟨*proof*⟩

8.5 Images of Functions

lemma *image-lam*: $C \subseteq A \Longrightarrow (\lambda x{\in}A.\ b(x))\ ``\ C = \{b(x).\ x \in C\}$
⟨*proof*⟩

lemma *Repfun-function-if*:
function(*f*)
$\Longrightarrow \{f`x.\ x \in C\} = (\mathit{if}\ C \subseteq \mathit{domain}(f)\ \mathit{then}\ f``C\ \mathit{else}\ \mathit{cons}(0, f``C))$
⟨*proof*⟩

lemma *image-function*:
$[\![\mathit{function}(f);\ \ C \subseteq \mathit{domain}(f)]\!] \Longrightarrow f``C = \{f`x.\ x \in C\}$
⟨*proof*⟩

lemma *image-fun*: $[\![f \in \mathit{Pi}(A,B);\ \ C \subseteq A]\!] \Longrightarrow f``C = \{f`x.\ x \in C\}$
⟨*proof*⟩

lemma *image-eq-UN*:
assumes *f*: $f \in \mathit{Pi}(A,B)\ C \subseteq A$ **shows** $f``C = (\bigcup x \in C.\ \{f\ `\ x\})$
⟨*proof*⟩

lemma *Pi-image-cons*:
$[\![f \in \mathit{Pi}(A,B);\ \ x \in A]\!] \Longrightarrow f\ ``\ \mathit{cons}(x,y) = \mathit{cons}(f`x,\ f``y)$
⟨*proof*⟩

8.6 Properties of *restrict*(*f*, *A*)

lemma *restrict-subset*: $\mathit{restrict}(f,A) \subseteq f$
⟨*proof*⟩

lemma *function-restrictI*:
$\mathit{function}(f) \Longrightarrow \mathit{function}(\mathit{restrict}(f,A))$
⟨*proof*⟩

lemma *restrict-type2*: $[\![f \in \mathit{Pi}(C,B);\ \ A<=C]\!] \Longrightarrow \mathit{restrict}(f,A) \in \mathit{Pi}(A,B)$
⟨*proof*⟩

lemma *restrict*: $\mathit{restrict}(f,A)\ `\ a = (\mathit{if}\ a \in A\ \mathit{then}\ f`a\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *restrict-empty* [*simp*]: $\mathit{restrict}(f,0) = 0$
⟨*proof*⟩

lemma *restrict-iff*: $z \in \mathit{restrict}(r,A) \longleftrightarrow z \in r \wedge (\exists x \in A.\ \exists y.\ z = \langle x,\ y\rangle)$
⟨*proof*⟩

lemma *restrict-restrict* [*simp*]:
$\mathit{restrict}(\mathit{restrict}(r,A),B) = \mathit{restrict}(r,\ A \cap B)$
⟨*proof*⟩

lemma *domain-restrict* [*simp*]: $\mathit{domain}(\mathit{restrict}(f,C)) = \mathit{domain}(f) \cap C$
⟨*proof*⟩

lemma *restrict-idem*: $f \subseteq Sigma(A,B) \Longrightarrow restrict(f,A) = f$
⟨*proof*⟩

lemma *domain-restrict-idem*:
$[\![domain(r) \subseteq A;\ relation(r)]\!] \Longrightarrow restrict(r,A) = r$
⟨*proof*⟩

lemma *domain-restrict-lam* [*simp*]: $domain(restrict(Lambda(A,f),C)) = A \cap C$
⟨*proof*⟩

lemma *restrict-if* [*simp*]: $restrict(f,A)\ `\ a = (if\ a \in A\ then\ f`a\ else\ 0)$
⟨*proof*⟩

lemma *restrict-lam-eq*:
$A<=C \Longrightarrow restrict(\lambda x{\in}C.\ b(x),\ A) = (\lambda x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *fun-cons-restrict-eq*:
$f \in cons(a,\ b) -> B \Longrightarrow f = cons(<a,\ f\ `\ a>,\ restrict(f,\ b))$
⟨*proof*⟩

8.7 Unions of Functions

lemma *function-Union*:
$[\![\forall x{\in}S.\ function(x);$
$\forall x{\in}S.\ \forall y{\in}S.\ x<=y \mid y<=x]\!]$
$\Longrightarrow function(\bigcup(S))$
⟨*proof*⟩

lemma *fun-Union*:
$[\![\forall f{\in}S.\ \exists C\ D.\ f \in C->D;$
$\forall f{\in}S.\ \forall y{\in}S.\ f<=y \mid y<=f]\!] \Longrightarrow$
$\bigcup(S) \in domain(\bigcup(S)) -> range(\bigcup(S))$
⟨*proof*⟩

lemma *gen-relation-Union*:
$(\bigwedge f.\ f{\in}F \Longrightarrow relation(f)) \Longrightarrow relation(\bigcup(F))$
⟨*proof*⟩

lemmas *Un-rls* = *Un-subset-iff SUM-Un-distrib1 prod-Un-distrib2*
subset-trans [*OF - Un-upper1*]
subset-trans [*OF - Un-upper2*]

lemma *fun-disjoint-Un*:
$[\![f \in A->B;\ \ g \in C->D;\ \ A \cap C = 0]\!]$
$\Longrightarrow (f \cup g) \in (A \cup C) -> (B \cup D)$

⟨*proof*⟩

lemma *fun-disjoint-apply1*: $a \notin domain(g) \Longrightarrow (f \cup g)`a = f`a$
⟨*proof*⟩

lemma *fun-disjoint-apply2*: $c \notin domain(f) \Longrightarrow (f \cup g)`c = g`c$
⟨*proof*⟩

8.8 Domain and Range of a Function or Relation

lemma *domain-of-fun*: $f \in Pi(A,B) \Longrightarrow domain(f)=A$
⟨*proof*⟩

lemma *apply-rangeI*: $[\![f \in Pi(A,B);\ \ a \in A]\!] \Longrightarrow f`a \in range(f)$
⟨*proof*⟩

lemma *range-of-fun*: $f \in Pi(A,B) \Longrightarrow f \in A->range(f)$
⟨*proof*⟩

8.9 Extensions of Functions

lemma *fun-extend*:
$[\![f \in A->B;\ \ c\notin A]\!] \Longrightarrow cons(\langle c,b\rangle,f) \in cons(c,A) -> cons(b,B)$
⟨*proof*⟩

lemma *fun-extend3*:
$[\![f \in A->B;\ \ c\notin A;\ \ b \in B]\!] \Longrightarrow cons(\langle c,b\rangle,f) \in cons(c,A) -> B$
⟨*proof*⟩

lemma *extend-apply*:
$c \notin domain(f) \Longrightarrow cons(\langle c,b\rangle,f)`a = (if\ a=c\ then\ b\ else\ f`a)$
⟨*proof*⟩

lemma *fun-extend-apply* [*simp*]:
$[\![f \in A->B;\ \ c\notin A]\!] \Longrightarrow cons(\langle c,b\rangle,f)`a = (if\ a=c\ then\ b\ else\ f`a)$
⟨*proof*⟩

lemmas *singleton-apply* = *apply-equality* [*OF singletonI singleton-fun*, *simp*]

lemma *cons-fun-eq*:
$c \notin A \Longrightarrow cons(c,A) -> B = (\bigcup f \in A->B.\ \bigcup b\in B.\ \{cons(\langle c,b\rangle,\ f)\})$
⟨*proof*⟩

lemma *succ-fun-eq*: $succ(n) -> B = (\bigcup f \in n->B.\ \bigcup b\in B.\ \{cons(\langle n,b\rangle,\ f)\})$
⟨*proof*⟩

8.10 Function Updates

definition
update :: $[i,i,i] \Rightarrow i$ **where**
$update(f,a,b) \equiv \lambda x{\in}cons(a,\ domain(f)).\ if(x{=}a,\ b,\ f`x)$

nonterminal *updbinds* **and** *updbind*

syntax

-updbind :: $[i,\ i] \Rightarrow updbind$ ($\langle(2\text{-} :=/\ \text{-})\rangle$)
:: $updbind \Rightarrow updbinds$ ($\langle\text{-}\rangle$)
-updbinds :: $[updbind,\ updbinds] \Rightarrow updbinds$ ($\langle\text{-},/\ \text{-}\rangle$)
-Update :: $[i,\ updbinds] \Rightarrow i$ ($\langle\text{-}/'((\text{-})')\rangle$ $[900,0]$ 900)

translations
-Update $(f,\ \text{-}updbinds(b,bs))$ $==$ *-Update* $(\text{-}Update(f,b),\ bs)$
$f(x{:=}y)$ $==$ $CONST\ update(f,x,y)$

lemma *update-apply* [*simp*]: $f(x{:=}y)\ `\ z = (if\ z{=}x\ then\ y\ else\ f`z)$
⟨*proof*⟩

lemma *update-idem*: $[\![f`x = y;\ \ f \in Pi(A,B);\ \ x \in A]\!] \Longrightarrow f(x{:=}y) = f$
⟨*proof*⟩

declare *refl* [*THEN update-idem*, *simp*]

lemma *domain-update* [*simp*]: $domain(f(x{:=}y)) = cons(x,\ domain(f))$
⟨*proof*⟩

lemma *update-type*: $[\![f \in Pi(A,B);\ \ x \in A;\ \ y \in B(x)]\!] \Longrightarrow f(x{:=}y) \in Pi(A,\ B)$
⟨*proof*⟩

8.11 Monotonicity Theorems

8.11.1 Replacement in its Various Forms

lemma *Replace-mono*: $A{<=}B \Longrightarrow Replace(A,P) \subseteq Replace(B,P)$
⟨*proof*⟩

lemma *RepFun-mono*: $A{<=}B \Longrightarrow \{f(x).\ x \in A\} \subseteq \{f(x).\ x \in B\}$
⟨*proof*⟩

lemma *Pow-mono*: $A{<=}B \Longrightarrow Pow(A) \subseteq Pow(B)$
⟨*proof*⟩

lemma *Union-mono*: $A<=B \Longrightarrow \bigcup(A) \subseteq \bigcup(B)$
⟨*proof*⟩

lemma *UN-mono*:
$[\![A<=C;\ \bigwedge x.\ x \in A \Longrightarrow B(x)<=D(x)]\!] \Longrightarrow (\bigcup x{\in}A.\ B(x)) \subseteq (\bigcup x{\in}C.\ D(x))$
⟨*proof*⟩

lemma *Inter-anti-mono*: $[\![A<=B;\ A\neq 0]\!] \Longrightarrow \bigcap(B) \subseteq \bigcap(A)$
⟨*proof*⟩

lemma *cons-mono*: $C<=D \Longrightarrow cons(a,C) \subseteq cons(a,D)$
⟨*proof*⟩

lemma *Un-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A \cup B \subseteq C \cup D$
⟨*proof*⟩

lemma *Int-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A \cap B \subseteq C \cap D$
⟨*proof*⟩

lemma *Diff-mono*: $[\![A<=C;\ D<=B]\!] \Longrightarrow A-B \subseteq C-D$
⟨*proof*⟩

8.11.2 Standard Products, Sums and Function Spaces

lemma *Sigma-mono* [*rule-format*]:
$[\![A<=C;\ \bigwedge x.\ x \in A \longrightarrow B(x) \subseteq D(x)]\!] \Longrightarrow Sigma(A,B) \subseteq Sigma(C,D)$
⟨*proof*⟩

lemma *sum-mono*: $[\![A<=C;\ B<=D]\!] \Longrightarrow A+B \subseteq C+D$
⟨*proof*⟩

lemma *Pi-mono*: $B<=C \Longrightarrow A->B \subseteq A->C$
⟨*proof*⟩

lemma *lam-mono*: $A<=B \Longrightarrow Lambda(A,c) \subseteq Lambda(B,c)$
⟨*proof*⟩

8.11.3 Converse, Domain, Range, Field

lemma *converse-mono*: $r<=s \Longrightarrow converse(r) \subseteq converse(s)$
⟨*proof*⟩

lemma *domain-mono*: $r<=s \Longrightarrow domain(r)<=domain(s)$
⟨*proof*⟩

lemmas *domain-rel-subset* = *subset-trans* [*OF domain-mono domain-subset*]

lemma *range-mono*: $r<=s \Longrightarrow range(r)<=range(s)$
⟨*proof*⟩

lemmas *range-rel-subset* = *subset-trans* [*OF range-mono range-subset*]

lemma *field-mono*: $r<=s \Longrightarrow field(r)<=field(s)$
⟨*proof*⟩

lemma *field-rel-subset*: $r \subseteq A*A \Longrightarrow field(r) \subseteq A$
⟨*proof*⟩

8.11.4 Images

lemma *image-pair-mono*:
$[\![\bigwedge x\ y.\ \langle x,y\rangle:r \Longrightarrow \langle x,y\rangle:s;\ \ A<=B]\!] \Longrightarrow r``A \subseteq s``B$
⟨*proof*⟩

lemma *vimage-pair-mono*:
$[\![\bigwedge x\ y.\ \langle x,y\rangle:r \Longrightarrow \langle x,y\rangle:s;\ \ A<=B]\!] \Longrightarrow r-``A \subseteq s-``B$
⟨*proof*⟩

lemma *image-mono*: $[\![r<=s;\ \ A<=B]\!] \Longrightarrow r``A \subseteq s``B$
⟨*proof*⟩

lemma *vimage-mono*: $[\![r<=s;\ \ A<=B]\!] \Longrightarrow r-``A \subseteq s-``B$
⟨*proof*⟩

lemma *Collect-mono*:
$[\![A<=B;\ \ \bigwedge x.\ x \in A \Longrightarrow P(x) \longrightarrow Q(x)]\!] \Longrightarrow Collect(A,P) \subseteq Collect(B,Q)$
⟨*proof*⟩

lemmas *basic-monos* = *subset-refl imp-refl disj-mono conj-mono ex-mono*
Collect-mono Part-mono in-mono

lemma *bex-image-simp*:
$[\![f \in Pi(X,\ Y);\ A \subseteq X]\!] \Longrightarrow (\exists\, x{\in}f``A.\ P(x)) \longleftrightarrow (\exists\, x{\in}A.\ P(f`x))$
⟨*proof*⟩

lemma *ball-image-simp*:
$[\![f \in Pi(X,\ Y);\ A \subseteq X]\!] \Longrightarrow (\forall\, x{\in}f``A.\ P(x)) \longleftrightarrow (\forall\, x{\in}A.\ P(f`x))$
⟨*proof*⟩

end

9 Quine-Inspired Ordered Pairs and Disjoint Sums

theory *QPair* **imports** *Sum func* **begin**

For non-well-founded data structures in ZF. Does not precisely follow Quine's construction. Thanks to Thomas Forster for suggesting this approach!

W. V. Quine, On Ordered Pairs and Relations, in Selected Logic Papers, 1966.

definition
  QPair :: [*i*, *i*] ⇒ *i* (‹<(-;/ -)>›) **where**
    <*a*;*b*> ≡ *a*+*b*

definition
  qfst :: *i* ⇒ *i* **where**
    qfst(*p*) ≡ *THE a*. ∃ *b*. *p*=<*a*;*b*>

definition
  qsnd :: *i* ⇒ *i* **where**
    qsnd(*p*) ≡ *THE b*. ∃ *a*. *p*=<*a*;*b*>

definition
  qsplit :: [[*i*, *i*] ⇒ *'a*, *i*] ⇒ *'a*::{} **where**
    qsplit(*c*,*p*) ≡ *c*(*qfst*(*p*), *qsnd*(*p*))

definition
  qconverse :: *i* ⇒ *i* **where**
    qconverse(*r*) ≡ {*z*. *w* ∈ *r*, ∃ *x y*. *w*=<*x*;*y*> ∧ *z*=<*y*;*x*>}

definition
  QSigma :: [*i*, *i* ⇒ *i*] ⇒ *i* **where**
    QSigma(*A*,*B*) ≡ ⋃*x*∈*A*. ⋃*y*∈*B*(*x*). {<*x*;*y*>}

syntax
  -QSUM :: [*idt*, *i*, *i*] ⇒ *i* (‹(*3QSUM* - ∈ -./ -)› *10*)
translations
  QSUM x ∈ *A*. *B* => *CONST QSigma*(*A*, λ*x*. *B*)

abbreviation
  qprod (**infixr** ‹<*>› *80*) **where**
  A <*> *B* ≡ *QSigma*(*A*, λ-. *B*)

definition
  qsum :: [*i*,*i*]⇒*i* (**infixr** ‹<+>› *65*) **where**
    A <+> *B* ≡ ({*0*} <*> *A*) ∪ ({*1*} <*> *B*)

definition
  QInl :: *i*⇒*i* **where**
    QInl(*a*) ≡ <*0*;*a*>

definition
QInr :: $i \Rightarrow i$ **where**
QInr(*b*) $\equiv <1;b>$

definition
qcase :: $[i \Rightarrow i,\ i \Rightarrow i,\ i] \Rightarrow i$ **where**
qcase(*c*,*d*) $\equiv$ *qsplit*($\lambda y\ z.$ *cond*(y, *d*(z), *c*(z)))

9.1 Quine ordered pairing

lemma *QPair-empty* [*simp*]: $<0;0> = 0$
⟨*proof*⟩

lemma *QPair-iff* [*simp*]: $<a;b> = <c;d> \longleftrightarrow a=c \wedge b=d$
⟨*proof*⟩

lemmas *QPair-inject* = *QPair-iff* [*THEN iffD1*, *THEN conjE*, *elim*!]

lemma *QPair-inject1*: $<a;b> = <c;d> \Longrightarrow a=c$
⟨*proof*⟩

lemma *QPair-inject2*: $<a;b> = <c;d> \Longrightarrow b=d$
⟨*proof*⟩

9.1.1 QSigma: Disjoint union of a family of sets Generalizes Cartesian product

lemma *QSigmaI* [*intro*!]: $[\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow <a;b> \in QSigma(A,B)$
⟨*proof*⟩

lemma *QSigmaE* [*elim*!]:
$[\![c \in QSigma(A,B);$
$\bigwedge x\ y.[\![x \in A;\ \ y \in B(x);\ \ c=<x;y>]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *QSigmaE2* [*elim*!]:
$[\![<a;b>: QSigma(A,B);\ [\![a \in A;\ \ b \in B(a)]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *QSigmaD1*: $<a;b> \in QSigma(A,B) \Longrightarrow a \in A$
⟨*proof*⟩

lemma *QSigmaD2*: $<a;b> \in QSigma(A,B) \Longrightarrow b \in B(a)$
⟨*proof*⟩

lemma *QSigma-cong*:

$[\![A{=}A';\ \bigwedge x.\ x \in A' \Longrightarrow B(x){=}B'(x)]\!] \Longrightarrow$
$QSigma(A,B) = QSigma(A',B')$
⟨*proof*⟩

lemma *QSigma-empty1* [*simp*]: $QSigma(0,B) = 0$
⟨*proof*⟩

lemma *QSigma-empty2* [*simp*]: $A <*> 0 = 0$
⟨*proof*⟩

9.1.2 Projections: qfst, qsnd

lemma *qfst-conv* [*simp*]: $qfst(<a;b>) = a$
⟨*proof*⟩

lemma *qsnd-conv* [*simp*]: $qsnd(<a;b>) = b$
⟨*proof*⟩

lemma *qfst-type* [*TC*]: $p \in QSigma(A,B) \Longrightarrow qfst(p) \in A$
⟨*proof*⟩

lemma *qsnd-type* [*TC*]: $p \in QSigma(A,B) \Longrightarrow qsnd(p) \in B(qfst(p))$
⟨*proof*⟩

lemma *QPair-qfst-qsnd-eq*: $a \in QSigma(A,B) \Longrightarrow <qfst(a);\ qsnd(a)> = a$
⟨*proof*⟩

9.1.3 Eliminator: qsplit

lemma *qsplit* [*simp*]: $qsplit(\lambda x\ y.\ c(x,y),\ <a;b>) \equiv c(a,b)$
⟨*proof*⟩

lemma *qsplit-type* [*elim*!]:
$[\![p \in QSigma(A,B);$
$\bigwedge x\ y.[\![x \in A;\ y \in B(x)]\!] \Longrightarrow c(x,y){:}C(<x;y>)$
$]\!] \Longrightarrow qsplit(\lambda x\ y.\ c(x,y),\ p) \in C(p)$
⟨*proof*⟩

lemma *expand-qsplit*:
$u \in A<*>B \Longrightarrow R(qsplit(c,u)) \longleftrightarrow (\forall\, x{\in}A.\ \forall\, y{\in}B.\ u = <x;y> \longrightarrow R(c(x,y)))$
⟨*proof*⟩

9.1.4 qsplit for predicates: result type o

lemma *qsplitI*: $R(a,b) \Longrightarrow qsplit(R,\ <a;b>)$
⟨*proof*⟩

lemma *qsplitE*:

⟦*qsplit*(*R*,*z*); *z* ∈ *QSigma*(*A*,*B*);
⋀*x y*. ⟦*z* = <*x*;*y*>; *R*(*x*,*y*)⟧ ⟹ *P*
⟧ ⟹ *P*
⟨*proof*⟩

lemma *qsplitD*: *qsplit*(*R*,<*a*;*b*>) ⟹ *R*(*a*,*b*)
⟨*proof*⟩

9.1.5 qconverse

lemma *qconverseI* [*intro*!]: <*a*;*b*>:*r* ⟹ <*b*;*a*>:*qconverse*(*r*)
⟨*proof*⟩

lemma *qconverseD* [*elim*!]: <*a*;*b*> ∈ *qconverse*(*r*) ⟹ <*b*;*a*> ∈ *r*
⟨*proof*⟩

lemma *qconverseE* [*elim*!]:
⟦*yx* ∈ *qconverse*(*r*);
⋀*x y*. ⟦*yx*=<*y*;*x*>; <*x*;*y*>:*r*⟧ ⟹ *P*
⟧ ⟹ *P*
⟨*proof*⟩

lemma *qconverse-qconverse*: *r*<=*QSigma*(*A*,*B*) ⟹ *qconverse*(*qconverse*(*r*)) = *r*
⟨*proof*⟩

lemma *qconverse-type*: *r* ⊆ *A* <*> *B* ⟹ *qconverse*(*r*) ⊆ *B* <*> *A*
⟨*proof*⟩

lemma *qconverse-prod*: *qconverse*(*A* <*> *B*) = *B* <*> *A*
⟨*proof*⟩

lemma *qconverse-empty*: *qconverse*(*0*) = *0*
⟨*proof*⟩

9.2 The Quine-inspired notion of disjoint sum

lemmas *qsum-defs* = *qsum-def QInl-def QInr-def qcase-def*

lemma *QInlI* [*intro*!]: *a* ∈ *A* ⟹ *QInl*(*a*) ∈ *A* <+> *B*
⟨*proof*⟩

lemma *QInrI* [*intro*!]: *b* ∈ *B* ⟹ *QInr*(*b*) ∈ *A* <+> *B*
⟨*proof*⟩

lemma *qsumE* [*elim*!]:
⟦*u* ∈ *A* <+> *B*;

$\bigwedge x.\ [\![x \in A;\ \ u{=}QInl(x)]\!] \Longrightarrow P;$
$\bigwedge y.\ [\![y \in B;\ \ u{=}QInr(y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *QInl-iff* [*iff*]: $QInl(a){=}QInl(b) \longleftrightarrow a{=}b$
⟨*proof*⟩

lemma *QInr-iff* [*iff*]: $QInr(a){=}QInr(b) \longleftrightarrow a{=}b$
⟨*proof*⟩

lemma *QInl-QInr-iff* [*simp*]: $QInl(a){=}QInr(b) \longleftrightarrow False$
⟨*proof*⟩

lemma *QInr-QInl-iff* [*simp*]: $QInr(b){=}QInl(a) \longleftrightarrow False$
⟨*proof*⟩

lemma *qsum-empty* [*simp*]: $0{<+>}0 = 0$
⟨*proof*⟩

lemmas *QInl-inject* = *QInl-iff* [*THEN iffD1*]
lemmas *QInr-inject* = *QInr-iff* [*THEN iffD1*]
lemmas *QInl-neq-QInr* = *QInl-QInr-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]
lemmas *QInr-neq-QInl* = *QInr-QInl-iff* [*THEN iffD1*, *THEN FalseE*, *elim*!]

lemma *QInlD*: $QInl(a)$: $A{<+>}B \Longrightarrow a \in A$
⟨*proof*⟩

lemma *QInrD*: $QInr(b)$: $A{<+>}B \Longrightarrow b \in B$
⟨*proof*⟩

lemma *qsum-iff*:
$u \in A <+> B \longleftrightarrow (\exists x.\ x \in A \wedge u{=}QInl(x)) \mid (\exists y.\ y \in B \wedge u{=}QInr(y))$
⟨*proof*⟩

lemma *qsum-subset-iff*: $A <+> B \subseteq C <+> D \longleftrightarrow A{<=}C \wedge B{<=}D$
⟨*proof*⟩

lemma *qsum-equal-iff*: $A <+> B = C <+> D \longleftrightarrow A{=}C \wedge B{=}D$
⟨*proof*⟩

9.2.1 Eliminator − qcase

lemma *qcase-QInl* [*simp*]: $qcase(c,\ d,\ QInl(a)) = c(a)$
⟨*proof*⟩

lemma *qcase-QInr* [*simp*]: $qcase(c,\ d,\ QInr(b)) = d(b)$
⟨*proof*⟩

lemma *qcase-type*:
$[\![u \in A <+> B;$
$\bigwedge x.\ x \in A \Longrightarrow c(x):\ C(QInl(x));$
$\bigwedge y.\ y \in B \Longrightarrow d(y):\ C(QInr(y))$
$]\!] \Longrightarrow qcase(c,d,u) \in C(u)$
⟨*proof*⟩

lemma *Part-QInl*: $Part(A <+> B,QInl) = \{QInl(x).\ x \in A\}$
⟨*proof*⟩

lemma *Part-QInr*: $Part(A <+> B,QInr) = \{QInr(y).\ y \in B\}$
⟨*proof*⟩

lemma *Part-QInr2*: $Part(A <+> B,\ \lambda x.\ QInr(h(x))) = \{QInr(y).\ y \in Part(B,h)\}$
⟨*proof*⟩

lemma *Part-qsum-equality*: $C \subseteq A <+> B \Longrightarrow Part(C,QInl) \cup Part(C,QInr) = C$
⟨*proof*⟩

9.2.2 Monotonicity

lemma *QPair-mono*: $[\![a<=c;\ \ b<=d]\!] \Longrightarrow <a;b> \subseteq <c;d>$
⟨*proof*⟩

lemma *QSigma-mono* [*rule-format*]:
$[\![A<=C;\ \ \forall x \in A.\ B(x) \subseteq D(x)]\!] \Longrightarrow QSigma(A,B) \subseteq QSigma(C,D)$
⟨*proof*⟩

lemma *QInl-mono*: $a<=b \Longrightarrow QInl(a) \subseteq QInl(b)$
⟨*proof*⟩

lemma *QInr-mono*: $a<=b \Longrightarrow QInr(a) \subseteq QInr(b)$
⟨*proof*⟩

lemma *qsum-mono*: $[\![A<=C;\ \ B<=D]\!] \Longrightarrow A <+> B \subseteq C <+> D$
⟨*proof*⟩

end

10 Injections, Surjections, Bijections, Composition

theory *Perm* **imports** *func* **begin**

definition

comp :: $[i,i]\Rightarrow i$ (**infixr** $\langle O\rangle$ *60*) **where**
$r\ O\ s \equiv \{xz \in domain(s)*range(r)\ .\ \exists x\ y\ z.\ xz=\langle x,z\rangle \wedge \langle x,y\rangle:s \wedge \langle y,z\rangle:r\}$

definition

id :: $i\Rightarrow i$ **where**
$id(A) \equiv (\lambda x\in A.\ x)$

definition

inj :: $[i,i]\Rightarrow i$ **where**
$inj(A,B) \equiv \{\ f \in A->B.\ \forall w\in A.\ \forall x\in A.\ f`w=f`x \longrightarrow w=x\}$

definition

surj :: $[i,i]\Rightarrow i$ **where**
$surj(A,B) \equiv \{\ f \in A->B\ .\ \forall y\in B.\ \exists x\in A.\ f`x=y\}$

definition

bij :: $[i,i]\Rightarrow i$ **where**
$bij(A,B) \equiv inj(A,B) \cap surj(A,B)$

10.1 Surjective Function Space

lemma *surj-is-fun*: $f \in surj(A,B) \Longrightarrow f \in A->B$
⟨*proof*⟩

lemma *fun-is-surj*: $f \in Pi(A,B) \Longrightarrow f \in surj(A,range(f))$
⟨*proof*⟩

lemma *surj-range*: $f \in surj(A,B) \Longrightarrow range(f)=B$
⟨*proof*⟩

A function with a right inverse is a surjection

lemma *f-imp-surjective*:
$[\![f \in A->B;\ \bigwedge y.\ y \in B \Longrightarrow d(y)\colon A;\ \bigwedge y.\ y \in B \Longrightarrow f`d(y) = y]\!]$
$\Longrightarrow f \in surj(A,B)$
⟨*proof*⟩

lemma *lam-surjective*:
$[\![\bigwedge x.\ x \in A \Longrightarrow c(x)\colon B;$
$\bigwedge y.\ y \in B \Longrightarrow d(y)\colon A;$

$\bigwedge y.\ y \in B \Longrightarrow c(d(y)) = y$
$\rrbracket \Longrightarrow (\lambda x{\in}A.\ c(x)) \in surj(A,B)$
⟨*proof*⟩

Cantor's theorem revisited

lemma *cantor-surj*: $f \notin surj(A,Pow(A))$
⟨*proof*⟩

10.2 Injective Function Space

lemma *inj-is-fun*: $f \in inj(A,B) \Longrightarrow f \in A->B$
⟨*proof*⟩

Good for dealing with sets of pairs, but a bit ugly in use [used in AC]

lemma *inj-equality*:
$\llbracket \langle a,b\rangle{:}f;\ \langle c,b\rangle{:}f;\ f \in inj(A,B) \rrbracket \Longrightarrow a{=}c$
⟨*proof*⟩

lemma *inj-apply-equality*: $\llbracket f \in inj(A,B);\ f`a{=}f`b;\ a \in A;\ b \in A \rrbracket \Longrightarrow a{=}b$
⟨*proof*⟩

A function with a left inverse is an injection

lemma *f-imp-injective*: $\llbracket f \in A->B;\ \forall x{\in}A.\ d(f`x){=}x \rrbracket \Longrightarrow f \in inj(A,B)$
⟨*proof*⟩

lemma *lam-injective*:
$\llbracket \bigwedge x.\ x \in A \Longrightarrow c(x){:}\ B;$
$\bigwedge x.\ x \in A \Longrightarrow d(c(x)) = x \rrbracket$
$\Longrightarrow (\lambda x{\in}A.\ c(x)) \in inj(A,B)$
⟨*proof*⟩

10.3 Bijections

lemma *bij-is-inj*: $f \in bij(A,B) \Longrightarrow f \in inj(A,B)$
⟨*proof*⟩

lemma *bij-is-surj*: $f \in bij(A,B) \Longrightarrow f \in surj(A,B)$
⟨*proof*⟩

lemma *bij-is-fun*: $f \in bij(A,B) \Longrightarrow f \in A->B$
⟨*proof*⟩

lemma *lam-bijective*:
$\llbracket \bigwedge x.\ x \in A \Longrightarrow c(x){:}\ B;$
$\bigwedge y.\ y \in B \Longrightarrow d(y){:}\ A;$
$\bigwedge x.\ x \in A \Longrightarrow d(c(x)) = x;$
$\bigwedge y.\ y \in B \Longrightarrow c(d(y)) = y$
$\rrbracket \Longrightarrow (\lambda x{\in}A.\ c(x)) \in bij(A,B)$
⟨*proof*⟩

lemma *RepFun-bijective*: $(\forall\, y{\in}x.\ \exists!y'.\ f(y') = f(y))$
$\Longrightarrow (\lambda z{\in}\{f(y).\ y \in x\}.\ THE\ y.\ f(y) = z) \in bij(\{f(y).\ y \in x\},\ x)$
⟨*proof*⟩

10.4 Identity Function

lemma *idI* [*intro*!]: $a \in A \Longrightarrow \langle a,a\rangle \in id(A)$
⟨*proof*⟩

lemma *idE* [*elim*!]: $[\![p \in id(A);\ \bigwedge x.[\![x \in A;\ p{=}\langle x,x\rangle]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *id-type*: $id(A) \in A{->}A$
⟨*proof*⟩

lemma *id-conv* [*simp*]: $x \in A \Longrightarrow id(A)\,`x = x$
⟨*proof*⟩

lemma *id-mono*: $A{<=}B \Longrightarrow id(A) \subseteq id(B)$
⟨*proof*⟩

lemma *id-subset-inj*: $A{<=}B \Longrightarrow id(A)$: $inj(A,B)$
⟨*proof*⟩

lemmas *id-inj* = *subset-refl* [*THEN id-subset-inj*]

lemma *id-surj*: $id(A)$: $surj(A,A)$
⟨*proof*⟩

lemma *id-bij*: $id(A)$: $bij(A,A)$
⟨*proof*⟩

lemma *subset-iff-id*: $A \subseteq B \longleftrightarrow id(A) \in A{->}B$
⟨*proof*⟩

id as the identity relation

lemma *id-iff* [*simp*]: $\langle x,y\rangle \in id(A) \longleftrightarrow x{=}y \wedge y \in A$
⟨*proof*⟩

10.5 Converse of a Function

lemma *inj-converse-fun*: $f \in inj(A,B) \Longrightarrow converse(f) \in range(f){->}A$
⟨*proof*⟩

Equations for converse(f)

The premises are equivalent to saying that f is injective...

lemma *left-inverse-lemma*:

$[\![f \in A->B;\ \ converse(f)\colon C->A;\ \ a \in A]\!] \Longrightarrow converse(f)\,`(f`a) = a$
⟨*proof*⟩

lemma *left-inverse* [*simp*]: $[\![f \in inj(A,B);\ \ a \in A]\!] \Longrightarrow converse(f)\,`(f`a) = a$
⟨*proof*⟩

lemma *left-inverse-eq*:
$[\![f \in inj(A,B);\ f\ `\ x = y;\ x \in A]\!] \Longrightarrow converse(f)\ `\ y = x$
⟨*proof*⟩

lemmas *left-inverse-bij* = *bij-is-inj* [*THEN left-inverse*]

lemma *right-inverse-lemma*:
$[\![f \in A->B;\ \ converse(f)\colon C->A;\ \ b \in C]\!] \Longrightarrow f`(converse(f)\,`b) = b$
⟨*proof*⟩

lemma *right-inverse* [*simp*]:
$[\![f \in inj(A,B);\ \ b \in range(f)]\!] \Longrightarrow f`(converse(f)\,`b) = b$
⟨*proof*⟩

lemma *right-inverse-bij*: $[\![f \in bij(A,B);\ \ b \in B]\!] \Longrightarrow f`(converse(f)\,`b) = b$
⟨*proof*⟩

10.6 Converses of Injections, Surjections, Bijections

lemma *inj-converse-inj*: $f \in inj(A,B) \Longrightarrow converse(f)\colon inj(range(f),\ A)$
⟨*proof*⟩

lemma *inj-converse-surj*: $f \in inj(A,B) \Longrightarrow converse(f)\colon surj(range(f),\ A)$
⟨*proof*⟩

Adding this as an intro! rule seems to cause looping

lemma *bij-converse-bij* [*TC*]: $f \in bij(A,B) \Longrightarrow converse(f)\colon bij(B,A)$
⟨*proof*⟩

10.7 Composition of Two Relations

The inductive definition package could derive these theorems for $r\ O\ s$

lemma *compI* [*intro*]: $[\![\langle a,b\rangle{:}s;\ \langle b,c\rangle{:}r]\!] \Longrightarrow \langle a,c\rangle \in r\ O\ s$
⟨*proof*⟩

lemma *compE* [*elim*!]:
$[\![xz \in r\ O\ s;$
$\ \ \bigwedge x\ y\ z.\ [\![xz{=}\langle x,z\rangle;\ \ \langle x,y\rangle{:}s;\ \ \langle y,z\rangle{:}r]\!] \Longrightarrow P]\!]$
$\ \ \Longrightarrow P$
⟨*proof*⟩

lemma *compEpair*:

$[\![\langle a,c\rangle \in r\ O\ s;$
$\bigwedge y.\ [\![\langle a,y\rangle{:}s;\ \langle y,c\rangle{:}r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
⟨*proof*⟩

lemma *converse-comp*: $converse(R\ O\ S) = converse(S)\ O\ converse(R)$
⟨*proof*⟩

10.8 Domain and Range – see Suppes, Section 3.1

Boyer et al., Set Theory in First-Order Logic, JAR 2 (1986), 287-327

lemma *range-comp*: $range(r\ O\ s) \subseteq range(r)$
⟨*proof*⟩

lemma *range-comp-eq*: $domain(r) \subseteq range(s) \Longrightarrow range(r\ O\ s) = range(r)$
⟨*proof*⟩

lemma *domain-comp*: $domain(r\ O\ s) \subseteq domain(s)$
⟨*proof*⟩

lemma *domain-comp-eq*: $range(s) \subseteq domain(r) \Longrightarrow domain(r\ O\ s) = domain(s)$
⟨*proof*⟩

lemma *image-comp*: $(r\ O\ s)\text{``}A = r\text{``}(s\text{``}A)$
⟨*proof*⟩

lemma *inj-inj-range*: $f \in inj(A,B) \Longrightarrow f \in inj(A,range(f))$
⟨*proof*⟩

lemma *inj-bij-range*: $f \in inj(A,B) \Longrightarrow f \in bij(A,range(f))$
⟨*proof*⟩

10.9 Other Results

lemma *comp-mono*: $[\![r'{<=}r;\ s'{<=}s]\!] \Longrightarrow (r'\ O\ s') \subseteq (r\ O\ s)$
⟨*proof*⟩

composition preserves relations

lemma *comp-rel*: $[\![s{<=}A{*}B;\ \ r{<=}B{*}C]\!] \Longrightarrow (r\ O\ s) \subseteq A{*}C$
⟨*proof*⟩

associative law for composition

lemma *comp-assoc*: $(r\ O\ s)\ O\ t = r\ O\ (s\ O\ t)$
⟨*proof*⟩

lemma *left-comp-id*: $r{<=}A{*}B \Longrightarrow id(B)\ O\ r = r$
⟨*proof*⟩

lemma *right-comp-id*: $r<=A*B \Longrightarrow r\ O\ id(A) = r$
⟨*proof*⟩

10.10 Composition Preserves Functions, Injections, and Surjections

lemma *comp-function*: $[\![function(g);\ function(f)]\!] \Longrightarrow function(f\ O\ g)$
⟨*proof*⟩

Don't think the premises can be weakened much

lemma *comp-fun*: $[\![g \in A->B;\ f \in B->C]\!] \Longrightarrow (f\ O\ g) \in A->C$
⟨*proof*⟩

lemma *comp-fun-apply* [*simp*]:
$[\![g \in A->B;\ a \in A]\!] \Longrightarrow (f\ O\ g)`a = f`(g`a)$
⟨*proof*⟩

Simplifies compositions of lambda-abstractions

lemma *comp-lam*:
$[\![\bigwedge x.\ x \in A \Longrightarrow b(x): B]\!]$
$\Longrightarrow (\lambda y{\in}B.\ c(y))\ O\ (\lambda x{\in}A.\ b(x)) = (\lambda x{\in}A.\ c(b(x)))$
⟨*proof*⟩

lemma *comp-inj*:
$[\![g \in inj(A,B);\ f \in inj(B,C)]\!] \Longrightarrow (f\ O\ g) \in inj(A,C)$
⟨*proof*⟩

lemma *comp-surj*:
$[\![g \in surj(A,B);\ f \in surj(B,C)]\!] \Longrightarrow (f\ O\ g) \in surj(A,C)$
⟨*proof*⟩

lemma *comp-bij*:
$[\![g \in bij(A,B);\ f \in bij(B,C)]\!] \Longrightarrow (f\ O\ g) \in bij(A,C)$
⟨*proof*⟩

10.11 Dual Properties of *inj* and *surj*

Useful for proofs from D Pastre. Automatic theorem proving in set theory. Artificial Intelligence, 10:1–27, 1978.

lemma *comp-mem-injD1*:
$[\![(f\ O\ g): inj(A,C);\ g \in A->B;\ f \in B->C]\!] \Longrightarrow g \in inj(A,B)$
⟨*proof*⟩

lemma *comp-mem-injD2*:
$[\![(f\ O\ g): inj(A,C);\ g \in surj(A,B);\ f \in B->C]\!] \Longrightarrow f \in inj(B,C)$

⟨*proof*⟩

lemma *comp-mem-surjD1*:
⟦(*f O g*): *surj*(*A*,*C*); *g* ∈ *A*−>*B*; *f* ∈ *B*−>*C*⟧ ⟹ *f* ∈ *surj*(*B*,*C*)
⟨*proof*⟩

lemma *comp-mem-surjD2*:
⟦(*f O g*): *surj*(*A*,*C*); *g* ∈ *A*−>*B*; *f* ∈ *inj*(*B*,*C*)⟧ ⟹ *g* ∈ *surj*(*A*,*B*)
⟨*proof*⟩

10.11.1 Inverses of Composition

left inverse of composition; one inclusion is *f* ∈ *A* → *B* ⟹ *id*(*A*) ⊆ *converse*(*f*) *O f*

lemma *left-comp-inverse*: *f* ∈ *inj*(*A*,*B*) ⟹ *converse*(*f*) *O f* = *id*(*A*)
⟨*proof*⟩

right inverse of composition; one inclusion is *f* ∈ *A* → *B* ⟹ *f O converse*(*f*) ⊆ *id*(*B*)

lemma *right-comp-inverse*:
f ∈ *surj*(*A*,*B*) ⟹ *f O converse*(*f*) = *id*(*B*)
⟨*proof*⟩

10.11.2 Proving that a Function is a Bijection

lemma *comp-eq-id-iff*:
⟦*f* ∈ *A*−>*B*; *g* ∈ *B*−>*A*⟧ ⟹ *f O g* = *id*(*B*) ⟷ (∀ *y*∈*B*. *f*‘(*g*‘*y*)=*y*)
⟨*proof*⟩

lemma *fg-imp-bijective*:
⟦*f* ∈ *A*−>*B*; *g* ∈ *B*−>*A*; *f O g* = *id*(*B*); *g O f* = *id*(*A*)⟧ ⟹ *f* ∈ *bij*(*A*,*B*)
⟨*proof*⟩

lemma *nilpotent-imp-bijective*: ⟦*f* ∈ *A*−>*A*; *f O f* = *id*(*A*)⟧ ⟹ *f* ∈ *bij*(*A*,*A*)
⟨*proof*⟩

lemma *invertible-imp-bijective*:
⟦*converse*(*f*): *B*−>*A*; *f* ∈ *A*−>*B*⟧ ⟹ *f* ∈ *bij*(*A*,*B*)
⟨*proof*⟩

10.11.3 Unions of Functions

See similar theorems in func.thy

Theorem by KG, proof by LCP

lemma *inj-disjoint-Un*:
⟦*f* ∈ *inj*(*A*,*B*); *g* ∈ *inj*(*C*,*D*); *B* ∩ *D* = *0*⟧

$\Longrightarrow (\lambda a{\in}A \cup C.\ \mathit{if}\ a \in A\ \mathit{then}\ f`a\ \mathit{else}\ g`a) \in \mathit{inj}(A \cup C,\ B \cup D)$
⟨*proof*⟩

lemma *surj-disjoint-Un*:
$[\![f \in \mathit{surj}(A,B);\ \ g \in \mathit{surj}(C,D);\ \ A \cap C = 0]\!]$
$\Longrightarrow (f \cup g) \in \mathit{surj}(A \cup C,\ B \cup D)$
⟨*proof*⟩

A simple, high-level proof; the version for injections follows from it, using $f \in \mathit{inj}(A,\ B) \longleftrightarrow f \in \mathit{bij}(A,\ \mathit{range}(f))$

lemma *bij-disjoint-Un*:
$[\![f \in \mathit{bij}(A,B);\ \ g \in \mathit{bij}(C,D);\ \ A \cap C = 0;\ \ B \cap D = 0]\!]$
$\Longrightarrow (f \cup g) \in \mathit{bij}(A \cup C,\ B \cup D)$
⟨*proof*⟩

10.11.4 Restrictions as Surjections and Bijections

lemma *surj-image*:
$f \in \mathit{Pi}(A,B) \Longrightarrow f \in \mathit{surj}(A,\ f``A)$
⟨*proof*⟩

lemma *surj-image-eq*: $f \in \mathit{surj}(A,\ B) \Longrightarrow f``A = B$
⟨*proof*⟩

lemma *restrict-image* [*simp*]: $\mathit{restrict}(f,A)\ ``\ B = f\ ``\ (A \cap B)$
⟨*proof*⟩

lemma *restrict-inj*:
$[\![f \in \mathit{inj}(A,B);\ \ C{<=}A]\!] \Longrightarrow \mathit{restrict}(f,C){:}\ \mathit{inj}(C,B)$
⟨*proof*⟩

lemma *restrict-surj*: $[\![f \in \mathit{Pi}(A,B);\ \ C{<=}A]\!] \Longrightarrow \mathit{restrict}(f,C){:}\ \mathit{surj}(C,\ f``C)$
⟨*proof*⟩

lemma *restrict-bij*:
$[\![f \in \mathit{inj}(A,B);\ \ C{<=}A]\!] \Longrightarrow \mathit{restrict}(f,C){:}\ \mathit{bij}(C,\ f``C)$
⟨*proof*⟩

10.11.5 Lemmas for Ramsey's Theorem

lemma *inj-weaken-type*: $[\![f \in \mathit{inj}(A,B);\ \ B{<=}D]\!] \Longrightarrow f \in \mathit{inj}(A,D)$
⟨*proof*⟩

lemma *inj-succ-restrict*:
$[\![f \in \mathit{inj}(\mathit{succ}(m),\ A)]\!] \Longrightarrow \mathit{restrict}(f,m) \in \mathit{inj}(m,\ A{-}\{f`m\})$
⟨*proof*⟩

lemma *inj-extend*:

$[\![f \in inj(A,B);\ \ a\notin A;\ \ b\notin B]\!]$
$\Longrightarrow cons(\langle a,b\rangle,f) \in inj(cons(a,A),\ cons(b,B))$
$\langle proof\rangle$

end

11 Relations: Their General Properties and Transitive Closure

theory *Trancl* **imports** *Fixedpt Perm* **begin**

definition
refl :: $[i,i]\Rightarrow o$ **where**
$refl(A,r) \equiv (\forall x{\in}A.\ \langle x,x\rangle \in r)$

definition
irrefl :: $[i,i]\Rightarrow o$ **where**
$irrefl(A,r) \equiv \forall x{\in}A.\ \langle x,x\rangle \notin r$

definition
sym :: $i\Rightarrow o$ **where**
$sym(r) \equiv \forall x\ y.\ \langle x,y\rangle{:}\ r \longrightarrow \langle y,x\rangle{:}\ r$

definition
asym :: $i\Rightarrow o$ **where**
$asym(r) \equiv \forall x\ y.\ \langle x,y\rangle{:}r \longrightarrow \neg\ \langle y,x\rangle{:}r$

definition
antisym :: $i\Rightarrow o$ **where**
$antisym(r) \equiv \forall x\ y.\langle x,y\rangle{:}r \longrightarrow \langle y,x\rangle{:}r \longrightarrow x{=}y$

definition
trans :: $i\Rightarrow o$ **where**
$trans(r) \equiv \forall x\ y\ z.\ \langle x,y\rangle{:}\ r \longrightarrow \langle y,z\rangle{:}\ r \longrightarrow \langle x,z\rangle{:}\ r$

definition
trans-on :: $[i,i]\Rightarrow o$ $(\langle trans[\text{-}]'(\text{-}')\rangle)$ **where**
$trans[A](r) \equiv \forall x{\in}A.\ \forall y{\in}A.\ \forall z{\in}A.$
$\langle x,y\rangle{:}\ r \longrightarrow \langle y,z\rangle{:}\ r \longrightarrow \langle x,z\rangle{:}\ r$

definition
rtrancl :: $i\Rightarrow i$ $(\langle(\text{-}\widehat{}*)\rangle\ [100]\ 100)$ **where**
$r\widehat{}* \equiv lfp(field(r){*}field(r),\ \lambda s.\ id(field(r)) \cup (r\ O\ s))$

definition
trancl :: $i\Rightarrow i$ $(\langle(\text{-}\widehat{}+)\rangle\ [100]\ 100)$ **where**
$r\widehat{}+ \equiv r\ O\ r\widehat{}*$

definition
equiv :: $[i,i]\Rightarrow o$ **where**
$equiv(A,r) \equiv r \subseteq A*A \wedge refl(A,r) \wedge sym(r) \wedge trans(r)$

11.1 General properties of relations

11.1.1 irreflexivity

lemma *irreflI*:
$[\![\bigwedge x.\ x \in A \Longrightarrow \langle x,x\rangle \notin r]\!] \Longrightarrow irrefl(A,r)$
⟨*proof*⟩

lemma *irreflE*: $[\![irrefl(A,r);\ x \in A]\!] \Longrightarrow \langle x,x\rangle \notin r$
⟨*proof*⟩

11.1.2 symmetry

lemma *symI*:
$[\![\bigwedge x\ y.\langle x,y\rangle : r \Longrightarrow \langle y,x\rangle : r]\!] \Longrightarrow sym(r)$
⟨*proof*⟩

lemma *symE*: $[\![sym(r);\ \langle x,y\rangle : r]\!] \Longrightarrow \langle y,x\rangle : r$
⟨*proof*⟩

11.1.3 antisymmetry

lemma *antisymI*:
$[\![\bigwedge x\ y.[\![\langle x,y\rangle : r;\ \langle y,x\rangle : r]\!] \Longrightarrow x=y]\!] \Longrightarrow antisym(r)$
⟨*proof*⟩

lemma *antisymE*: $[\![antisym(r);\ \langle x,y\rangle : r;\ \langle y,x\rangle : r]\!] \Longrightarrow x=y$
⟨*proof*⟩

11.1.4 transitivity

lemma *transD*: $[\![trans(r);\ \langle a,b\rangle : r;\ \langle b,c\rangle : r]\!] \Longrightarrow \langle a,c\rangle : r$
⟨*proof*⟩

lemma *trans-onD*:
$[\![trans[A](r);\ \langle a,b\rangle : r;\ \langle b,c\rangle : r;\ a \in A;\ b \in A;\ c \in A]\!] \Longrightarrow \langle a,c\rangle : r$
⟨*proof*⟩

lemma *trans-imp-trans-on*: $trans(r) \Longrightarrow trans[A](r)$
⟨*proof*⟩

lemma *trans-on-imp-trans*: $[\![trans[A](r);\ r \subseteq A*A]\!] \Longrightarrow trans(r)$
⟨*proof*⟩

11.2 Transitive closure of a relation

lemma *rtrancl-bnd-mono*:

bnd-mono(*field*(*r*)∗*field*(*r*), λ*s*. *id*(*field*(*r*)) ∪ (*r* *O* *s*))
⟨*proof*⟩

lemma *rtrancl-mono*: *r*<=*s* ⟹ *r*^∗ ⊆ *s*^∗
⟨*proof*⟩

lemmas *rtrancl-unfold* =
rtrancl-bnd-mono [*THEN rtrancl-def* [*THEN def-lfp-unfold*]]

lemmas *rtrancl-type* = *rtrancl-def* [*THEN def-lfp-subset*]

lemma *relation-rtrancl*: *relation*(*r*^∗)
⟨*proof*⟩

lemma *rtrancl-refl*: ⟦*a* ∈ *field*(*r*)⟧ ⟹ ⟨*a*,*a*⟩ ∈ *r*^∗
⟨*proof*⟩

lemma *rtrancl-into-rtrancl*: ⟦⟨*a*,*b*⟩ ∈ *r*^∗; ⟨*b*,*c*⟩ ∈ *r*⟧ ⟹ ⟨*a*,*c*⟩ ∈ *r*^∗
⟨*proof*⟩

lemma *r-into-rtrancl*: ⟨*a*,*b*⟩ ∈ *r* ⟹ ⟨*a*,*b*⟩ ∈ *r*^∗
⟨*proof*⟩

lemma *r-subset-rtrancl*: *relation*(*r*) ⟹ *r* ⊆ *r*^∗
⟨*proof*⟩

lemma *rtrancl-field*: *field*(*r*^∗) = *field*(*r*)
⟨*proof*⟩

lemma *rtrancl-full-induct* [*case-names initial step*, *consumes 1*]:
⟦⟨*a*,*b*⟩ ∈ *r*^∗;
⋀*x*. *x* ∈ *field*(*r*) ⟹ *P*(⟨*x*,*x*⟩);
⋀*x y z*.⟦*P*(⟨*x*,*y*⟩); ⟨*x*,*y*⟩: *r*^∗; ⟨*y*,*z*⟩: *r*⟧ ⟹ *P*(⟨*x*,*z*⟩)⟧
⟹ *P*(⟨*a*,*b*⟩)
⟨*proof*⟩

lemma *rtrancl-induct* [*case-names initial step*, *induct set*: *rtrancl*]:

$[\![\langle a,b\rangle \in r\hat{}*;$
$P(a);$
$\bigwedge y\ z.[\![\langle a,y\rangle \in r\hat{}*;\ \langle y,z\rangle \in r;\ P(y)]\!] \Longrightarrow P(z)$
$]\!] \Longrightarrow P(b)$

⟨*proof*⟩

lemma *trans-rtrancl*: *trans*($r\hat{}*$)
⟨*proof*⟩

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*]

lemma *rtranclE*:
$[\![\langle a,b\rangle \in r\hat{}*;\ (a=b) \Longrightarrow P;$
$\bigwedge y.[\![\langle a,y\rangle \in r\hat{}*;\ \langle y,b\rangle \in r]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
⟨*proof*⟩

lemma *trans-trancl*: *trans*($r\hat{}+$)
⟨*proof*⟩

lemmas *trans-on-trancl* = *trans-trancl* [*THEN trans-imp-trans-on*]

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*]

lemma *trancl-into-rtrancl*: $\langle a,b\rangle \in r\hat{}+ \Longrightarrow \langle a,b\rangle \in r\hat{}*$
⟨*proof*⟩

lemma *r-into-trancl*: $\langle a,b\rangle \in r \Longrightarrow \langle a,b\rangle \in r\hat{}+$
⟨*proof*⟩

lemma *r-subset-trancl*: *relation*(r) $\Longrightarrow r \subseteq r\hat{}+$
⟨*proof*⟩

lemma *rtrancl-into-trancl1*: $[\![\langle a,b\rangle \in r\hat{}*;\ \langle b,c\rangle \in r]\!] \Longrightarrow \langle a,c\rangle \in r\hat{}+$
⟨*proof*⟩

lemma *rtrancl-into-trancl2*:
$[\![\langle a,b\rangle \in r;\ \langle b,c\rangle \in r\hat{}*]\!] \Longrightarrow \langle a,c\rangle \in r\hat{}+$
⟨*proof*⟩

lemma *trancl-induct* [*case-names initial step*, *induct set*: *trancl*]:
$[\![\langle a,b\rangle \in r\hat{}+;$
$\bigwedge y.\ [\![\langle a,y\rangle \in r]\!] \Longrightarrow P(y);$
$\bigwedge y\ z.[\![\langle a,y\rangle \in r\hat{}+;\ \langle y,z\rangle \in r;\ P(y)]\!] \Longrightarrow P(z)$
$]\!] \Longrightarrow P(b)$
⟨*proof*⟩

lemma *tranclE*:
$[\![\langle a,b\rangle \in r\hat{}+;$
$\langle a,b\rangle \in r \Longrightarrow P;$
$\bigwedge y.[\![\langle a,y\rangle \in r\hat{}+;\ \langle y,b\rangle \in r]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *trancl-type*: $r\hat{}+ \subseteq \mathit{field}(r)*\mathit{field}(r)$
⟨*proof*⟩

lemma *relation-trancl*: $\mathit{relation}(r\hat{}+)$
⟨*proof*⟩

lemma *trancl-subset-times*: $r \subseteq A * A \Longrightarrow r\hat{}+ \subseteq A * A$
⟨*proof*⟩

lemma *trancl-mono*: $r<=s \Longrightarrow r\hat{}+ \subseteq s\hat{}+$
⟨*proof*⟩

lemma *trancl-eq-r*: $[\![\mathit{relation}(r);\ \mathit{trans}(r)]\!] \Longrightarrow r\hat{}+ = r$
⟨*proof*⟩

lemma *rtrancl-idemp* [*simp*]: $(r\hat{}*)\hat{}* = r\hat{}*$
⟨*proof*⟩

lemma *rtrancl-subset*: $[\![R \subseteq S;\ S \subseteq R\hat{}*]\!] \Longrightarrow S\hat{}* = R\hat{}*$
⟨*proof*⟩

lemma *rtrancl-Un-rtrancl*:
$[\![\mathit{relation}(r);\ \mathit{relation}(s)]\!] \Longrightarrow (r\hat{}* \cup s\hat{}*)\hat{}* = (r \cup s)\hat{}*$
⟨*proof*⟩

lemma *rtrancl-converseD*: $\langle x,y\rangle$:*converse*$(r)\hat{}*$ $\Longrightarrow$ $\langle x,y\rangle$:*converse*$(r\hat{}*)$
⟨*proof*⟩

lemma *rtrancl-converseI*: $\langle x,y\rangle$:*converse*$(r\hat{}*)$ $\Longrightarrow$ $\langle x,y\rangle$:*converse*$(r)\hat{}*$
⟨*proof*⟩

lemma *rtrancl-converse*: *converse*$(r)\hat{}*$ = *converse*$(r\hat{}*)$
⟨*proof*⟩

lemma *trancl-converseD*: $\langle a,\ b\rangle$:*converse*$(r)\hat{}+$ $\Longrightarrow$ $\langle a,\ b\rangle$:*converse*$(r\hat{}+)$
⟨*proof*⟩

lemma *trancl-converseI*: $\langle x,y\rangle$:*converse*$(r\hat{}+)$ $\Longrightarrow$ $\langle x,y\rangle$:*converse*$(r)\hat{}+$
⟨*proof*⟩

lemma *trancl-converse*: *converse*$(r)\hat{}+$ = *converse*$(r\hat{}+)$
⟨*proof*⟩

lemma *converse-trancl-induct* [*case-names initial step*, *consumes 1*]:
$[\![\langle a,\ b\rangle$:$r\hat{}+;$ $\bigwedge y.\ \langle y,\ b\rangle$:$r \Longrightarrow P(y);$
$\bigwedge y\ z.\ [\![\langle y,\ z\rangle \in r;\ \langle z,\ b\rangle \in r\hat{}+;\ P(z)]\!] \Longrightarrow P(y)]\!]$
$\Longrightarrow P(a)$
⟨*proof*⟩

end

12 Well-Founded Recursion

theory *WF* **imports** *Trancl* **begin**

definition
wf :: $i \Rightarrow o$ **where**

$wf(r) \equiv \forall\, Z.\ Z=0 \mid (\exists\, x{\in}Z.\ \forall\, y.\ \langle y,x\rangle{:}r \longrightarrow \neg\ y \in Z)$

definition
wf-on :: $[i,i] \Rightarrow o$ (‹*wf*[-]'(-')›) **where**

$wf\text{-}on(A,r) \equiv wf(r \cap A{*}A)$

definition
is-recfun :: $[i,\ i,\ [i,i] \Rightarrow i,\ i] \Rightarrow o$ **where**
is-recfun$(r,a,H,f) \equiv (f = (\lambda x{\in}r-``\{a\}.\ H(x,\ restrict(f,\ r-``\{x\}))))$

definition
the-recfun :: $[i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
the-recfun$(r,a,H) \equiv (THE\ f.\ is\text{-}recfun(r,a,H,f))$

definition
wftrec :: $[i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
wftrec$(r,a,H) \equiv H(a,\ the\text{-}recfun(r,a,H))$

definition
wfrec :: $[i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**

wfrec$(r,a,H) \equiv wftrec(r^{+},\ a,\ \lambda x\ f.\ H(x,\ restrict(f,r-``\{x\})))$

definition
wfrec-on :: $[i,\ i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ (‹*wfrec*[-]'(-,-,-')›) **where**
wfrec$[A](r,a,H) \equiv wfrec(r \cap A*A,\ a,\ H)$

12.1 Well-Founded Relations

12.1.1 Equivalences between *wf* and *wf-on*

lemma *wf-imp-wf-on*: $wf(r) \Longrightarrow wf[A](r)$
⟨*proof*⟩

lemma *wf-on-imp-wf*: $[\![wf[A](r);\ r \subseteq A*A]\!] \Longrightarrow wf(r)$
⟨*proof*⟩

lemma *wf-on-field-imp-wf*: $wf[field(r)](r) \Longrightarrow wf(r)$
⟨*proof*⟩

lemma *wf-iff-wf-on-field*: $wf(r) \longleftrightarrow wf[field(r)](r)$
⟨*proof*⟩

lemma *wf-on-subset-A*: $[\![wf[A](r);\ \ B<=A]\!] \Longrightarrow wf[B](r)$
⟨*proof*⟩

lemma *wf-on-subset-r*: $[\![wf[A](r);\ s<=r]\!] \Longrightarrow wf[A](s)$
⟨*proof*⟩

lemma *wf-subset*: $[\![wf(s);\ r<=s]\!] \Longrightarrow wf(r)$
⟨*proof*⟩

12.1.2 Introduction Rules for *wf-on*

If every non-empty subset of A has an r-minimal element then we have $wf[A](r)$.

lemma *wf-onI*:
assumes *prem*: $\bigwedge Z\ u.\ [\![Z<=A;\ \ u \in Z;\ \ \forall x{\in}Z.\ \exists y{\in}Z.\ \langle y,x\rangle{:}r]\!] \Longrightarrow False$

shows $wf[A](r)$
⟨*proof*⟩

If r allows well-founded induction over A then we have $wf[A](r)$. Premise is equivalent to $\bigwedge B.\ \forall\, x{\in}A.\ (\forall\, y.\ \langle y,\ x\rangle \in r \longrightarrow y \in B) \longrightarrow x \in B \Longrightarrow A \subseteq B$

lemma *wf-onI2*:
assumes *prem*: $\bigwedge y\ B.\ [\![\forall\, x{\in}A.\ (\forall\, y{\in}A.\ \langle y,x\rangle{:}r \longrightarrow y \in B) \longrightarrow x \in B;\quad y \in A]\!] \Longrightarrow y \in B$
shows $wf[A](r)$
⟨*proof*⟩

12.1.3 Well-founded Induction

Consider the least z in $domain(r)$ such that $P(z)$ does not hold...

lemma *wf-induct-raw*:
$[\![wf(r);\ \bigwedge x.[\![\forall\, y.\ \langle y,x\rangle{:}\ r \longrightarrow P(y)]\!] \Longrightarrow P(x)]\!] \Longrightarrow P(a)$
⟨*proof*⟩

lemmas *wf-induct = wf-induct-raw* [*rule-format*, *consumes 1*, *case-names step*, *induct set*: *wf*]

The form of this rule is designed to match *wfI*

lemma *wf-induct2*:
$[\![wf(r);\ \ a \in A;\ \ field(r){<=}A;\ \bigwedge x.[\![x \in A;\ \forall\, y.\ \langle y,x\rangle{:}\ r \longrightarrow P(y)]\!] \Longrightarrow P(x)]\!] \Longrightarrow\ P(a)$
⟨*proof*⟩

lemma *field-Int-square*: $field(r \cap A{*}A) \subseteq A$
⟨*proof*⟩

lemma *wf-on-induct-raw* [*consumes 2*, *induct set*: *wf-on*]:
$[\![wf[A](r);\ \ a \in A;\ \bigwedge x.[\![x \in A;\ \forall\, y{\in}A.\ \langle y,x\rangle{:}\ r \longrightarrow P(y)]\!] \Longrightarrow P(x)]\!] \Longrightarrow\ P(a)$
⟨*proof*⟩

lemma *wf-on-induct* [*consumes 2*, *case-names step*, *induct set*: *wf-on*]:
$wf[A](r) \Longrightarrow a \in A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow \langle y,\ x\rangle \in r \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(a)$
⟨*proof*⟩

If r allows well-founded induction then we have $wf(r)$.

lemma *wfI*:
$[\![field(r){<=}A;$

$\bigwedge y\ B.\ [\![\forall x{\in}A.\ (\forall y{\in}A.\ \langle y,x\rangle{:}r \longrightarrow y \in B) \longrightarrow x \in B;\ \ y \in A]\!]$
$\quad \Longrightarrow y \in B]\!]$
$\Longrightarrow\ wf(r)$
⟨*proof*⟩

12.2 Basic Properties of Well-Founded Relations

lemma *wf-not-refl*: $wf(r) \Longrightarrow \langle a,a\rangle \notin r$
⟨*proof*⟩

lemma *wf-not-sym* [*rule-format*]: $wf(r) \Longrightarrow \forall x.\ \langle a,x\rangle{:}r \longrightarrow \langle x,a\rangle \notin r$
⟨*proof*⟩

lemmas *wf-asym* = *wf-not-sym* [*THEN swap*]

lemma *wf-on-not-refl*: $[\![wf[A](r);\ a \in A]\!] \Longrightarrow \langle a,a\rangle \notin r$
⟨*proof*⟩

lemma *wf-on-not-sym*:
$[\![wf[A](r);\ \ a \in A]\!] \Longrightarrow (\bigwedge b.\ b{\in}A \Longrightarrow \langle a,b\rangle{:}r \Longrightarrow \langle b,a\rangle{\notin}r)$
⟨*proof*⟩

lemma *wf-on-asym*:
$[\![wf[A](r);\ \neg Z \Longrightarrow \langle a,b\rangle \in r;$
$\langle b,a\rangle \notin r \Longrightarrow Z;\ \neg Z \Longrightarrow a \in A;\ \neg Z \Longrightarrow b \in A]\!] \Longrightarrow Z$
⟨*proof*⟩

lemma *wf-on-chain3*:
$[\![wf[A](r);\ \langle a,b\rangle{:}r;\ \langle b,c\rangle{:}r;\ \langle c,a\rangle{:}r;\ a \in A;\ b \in A;\ c \in A]\!] \Longrightarrow P$
⟨*proof*⟩

transitive closure of a WF relation is WF provided A is downward closed

lemma *wf-on-trancl*:
$[\![wf[A](r);\ \ r-``A \subseteq A]\!] \Longrightarrow wf[A](r\hat{}+)$
⟨*proof*⟩

lemma *wf-trancl*: $wf(r) \Longrightarrow wf(r\hat{}+)$
⟨*proof*⟩

$r -`` \{a\}$ is the set of everything under a in r

lemmas *underI* = *vimage-singleton-iff* [*THEN iffD2*]
lemmas *underD* = *vimage-singleton-iff* [*THEN iffD1*]

12.3 The Predicate *is-recfun*

lemma *is-recfun-type*: *is-recfun*$(r,a,H,f) \Longrightarrow f \in r-``\{a\}$ –> *range*(f)

⟨*proof*⟩

lemmas *is-recfun-imp-function* = *is-recfun-type* [*THEN fun-is-function*]

lemma *apply-recfun*:
⟦*is-recfun*(*r*,*a*,*H*,*f*); ⟨*x*,*a*⟩:*r*⟧ ⟹ *f*‘*x* = *H*(*x*, *restrict*(*f*,*r*−‘‘{*x*}))
⟨*proof*⟩

lemma *is-recfun-equal* [*rule-format*]:
⟦*wf*(*r*); *trans*(*r*); *is-recfun*(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*b*,*H*,*g*)⟧
⟹ ⟨*x*,*a*⟩:*r* ⟶ ⟨*x*,*b*⟩:*r* ⟶ *f*‘*x*=*g*‘*x*
⟨*proof*⟩

lemma *is-recfun-cut*:
⟦*wf*(*r*); *trans*(*r*);
is-recfun(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*b*,*H*,*g*); ⟨*b*,*a*⟩:*r*⟧
⟹ *restrict*(*f*, *r*−‘‘{*b*}) = *g*
⟨*proof*⟩

12.4 Recursion: Main Existence Lemma

lemma *is-recfun-functional*:
⟦*wf*(*r*); *trans*(*r*); *is-recfun*(*r*,*a*,*H*,*f*); *is-recfun*(*r*,*a*,*H*,*g*)⟧ ⟹ *f*=*g*
⟨*proof*⟩

lemma *the-recfun-eq*:
⟦*is-recfun*(*r*,*a*,*H*,*f*); *wf*(*r*); *trans*(*r*)⟧ ⟹ *the-recfun*(*r*,*a*,*H*) = *f*
⟨*proof*⟩

lemma *is-the-recfun*:
⟦*is-recfun*(*r*,*a*,*H*,*f*); *wf*(*r*); *trans*(*r*)⟧
⟹ *is-recfun*(*r*, *a*, *H*, *the-recfun*(*r*,*a*,*H*))
⟨*proof*⟩

lemma *unfold-the-recfun*:
⟦*wf*(*r*); *trans*(*r*)⟧ ⟹ *is-recfun*(*r*, *a*, *H*, *the-recfun*(*r*,*a*,*H*))
⟨*proof*⟩

12.5 Unfolding *wftrec*(*r*, *a*, *H*)

lemma *the-recfun-cut*:
⟦*wf*(*r*); *trans*(*r*); ⟨*b*,*a*⟩:*r*⟧
⟹ *restrict*(*the-recfun*(*r*,*a*,*H*), *r*−‘‘{*b*}) = *the-recfun*(*r*,*b*,*H*)
⟨*proof*⟩

lemma *wftrec*:
⟦*wf*(*r*); *trans*(*r*)⟧ ⟹
wftrec(*r*,*a*,*H*) = *H*(*a*, λ*x*∈*r*−‘‘{*a*}. *wftrec*(*r*,*x*,*H*))

⟨*proof*⟩

12.5.1 Removal of the Premise *trans*(*r*)

lemma *wfrec*:
wf(*r*) ⟹ *wfrec*(*r*,*a*,*H*) = *H*(*a*, λ*x*∈*r*−‘‘{*a*}. *wfrec*(*r*,*x*,*H*))
⟨*proof*⟩

lemma *def-wfrec*:
⟦⋀*x*. *h*(*x*)≡*wfrec*(*r*,*x*,*H*); *wf*(*r*)⟧ ⟹
h(*a*) = *H*(*a*, λ*x*∈*r*−‘‘{*a*}. *h*(*x*))
⟨*proof*⟩

lemma *wfrec-type*:
⟦*wf*(*r*); *a* ∈ *A*; *field*(*r*)<=*A*;
⋀*x u*. ⟦*x* ∈ *A*; *u* ∈ *Pi*(*r*−‘‘{*x*}, *B*)⟧ ⟹ *H*(*x*,*u*) ∈ *B*(*x*)
⟧ ⟹ *wfrec*(*r*,*a*,*H*) ∈ *B*(*a*)
⟨*proof*⟩

lemma *wfrec-on*:
⟦*wf*[*A*](*r*); *a* ∈ *A*⟧ ⟹
wfrec[*A*](*r*,*a*,*H*) = *H*(*a*, λ*x*∈(*r*−‘‘{*a*}) ∩ *A*. *wfrec*[*A*](*r*,*x*,*H*))
⟨*proof*⟩

Minimal-element characterization of well-foundedness

lemma *wf-eq-minimal*: *wf*(*r*) ⟷ (∀ *Q x*. *x* ∈ *Q* ⟶ (∃ *z*∈*Q*. ∀ *y*. ⟨*y*,*z*⟩:*r* ⟶ *y*∉*Q*))
⟨*proof*⟩

end

13 Transitive Sets and Ordinals

theory *Ordinal* **imports** *WF Bool equalities* **begin**

definition
Memrel :: *i*⇒*i* **where**
Memrel(*A*) ≡ {*z*∈*A*∗*A* . ∃ *x y*. *z*=⟨*x*,*y*⟩ ∧ *x*∈*y* }

definition
Transset :: *i*⇒*o* **where**
Transset(*i*) ≡ ∀ *x*∈*i*. *x*<=*i*

definition
Ord :: *i*⇒*o* **where**
Ord(*i*) ≡ *Transset*(*i*) ∧ (∀ *x*∈*i*. *Transset*(*x*))

definition
lt :: $[i,i] \Rightarrow o$ (**infixl** ‹<› 50) **where**
$i<j \equiv i\in j \wedge Ord(j)$

definition
Limit :: $i \Rightarrow o$ **where**
$Limit(i) \equiv Ord(i) \wedge 0<i \wedge (\forall y.\ y<i \longrightarrow succ(y)<i)$

abbreviation
le (**infixl** ‹≤› 50) **where**
$x \leq y \equiv x < succ(y)$

13.1 Rules for Transset

13.1.1 Three Neat Characterisations of Transset

lemma *Transset-iff-Pow*: $Transset(A) <-> A<=Pow(A)$
⟨*proof*⟩

lemma *Transset-iff-Union-succ*: $Transset(A) <-> \bigcup(succ(A)) = A$
⟨*proof*⟩

lemma *Transset-iff-Union-subset*: $Transset(A) <-> \bigcup(A) \subseteq A$
⟨*proof*⟩

13.1.2 Consequences of Downwards Closure

lemma *Transset-doubleton-D*:
$[\![Transset(C);\ \{a,b\}:\ C]\!] \Longrightarrow a\in C \wedge b\in C$
⟨*proof*⟩

lemma *Transset-Pair-D*:
$[\![Transset(C);\ \langle a,b\rangle\in C]\!] \Longrightarrow a\in C \wedge b\in C$
⟨*proof*⟩

lemma *Transset-includes-domain*:
$[\![Transset(C);\ A*B \subseteq C;\ b \in B]\!] \Longrightarrow A \subseteq C$
⟨*proof*⟩

lemma *Transset-includes-range*:
$[\![Transset(C);\ A*B \subseteq C;\ a \in A]\!] \Longrightarrow B \subseteq C$
⟨*proof*⟩

13.1.3 Closure Properties

lemma *Transset-0*: $Transset(0)$
⟨*proof*⟩

lemma *Transset-Un*:
$[\![Transset(i);\ \ Transset(j)]\!] \Longrightarrow Transset(i \cup j)$

⟨*proof*⟩

lemma *Transset-Int*:
⟦*Transset*(*i*); *Transset*(*j*)⟧ ⟹ *Transset*(*i* ∩ *j*)
⟨*proof*⟩

lemma *Transset-succ*: *Transset*(*i*) ⟹ *Transset*(*succ*(*i*))
⟨*proof*⟩

lemma *Transset-Pow*: *Transset*(*i*) ⟹ *Transset*(*Pow*(*i*))
⟨*proof*⟩

lemma *Transset-Union*: *Transset*(*A*) ⟹ *Transset*(⋃(*A*))
⟨*proof*⟩

lemma *Transset-Union-family*:
⟦⋀*i*. *i*∈*A* ⟹ *Transset*(*i*)⟧ ⟹ *Transset*(⋃(*A*))
⟨*proof*⟩

lemma *Transset-Inter-family*:
⟦⋀*i*. *i*∈*A* ⟹ *Transset*(*i*)⟧ ⟹ *Transset*(⋂(*A*))
⟨*proof*⟩

lemma *Transset-UN*:
(⋀*x*. *x* ∈ *A* ⟹ *Transset*(*B*(*x*))) ⟹ *Transset* (⋃*x*∈*A*. *B*(*x*))
⟨*proof*⟩

lemma *Transset-INT*:
(⋀*x*. *x* ∈ *A* ⟹ *Transset*(*B*(*x*))) ⟹ *Transset* (⋂*x*∈*A*. *B*(*x*))
⟨*proof*⟩

13.2 Lemmas for Ordinals

lemma *OrdI*:
⟦*Transset*(*i*); ⋀*x*. *x*∈*i* ⟹ *Transset*(*x*)⟧ ⟹ *Ord*(*i*)
⟨*proof*⟩

lemma *Ord-is-Transset*: *Ord*(*i*) ⟹ *Transset*(*i*)
⟨*proof*⟩

lemma *Ord-contains-Transset*:
⟦*Ord*(*i*); *j*∈*i*⟧ ⟹ *Transset*(*j*)
⟨*proof*⟩

lemma *Ord-in-Ord*: ⟦*Ord*(*i*); *j*∈*i*⟧ ⟹ *Ord*(*j*)
⟨*proof*⟩

lemma *Ord-in-Ord'*: $[\![j{\in}i;\ Ord(i)]\!] \Longrightarrow Ord(j)$
⟨*proof*⟩

lemmas *Ord-succD* = *Ord-in-Ord* [*OF - succI1*]

lemma *Ord-subset-Ord*: $[\![Ord(i);\ \ Transset(j);\ \ j{<=}i]\!] \Longrightarrow Ord(j)$
⟨*proof*⟩

lemma *OrdmemD*: $[\![j{\in}i;\ \ Ord(i)]\!] \Longrightarrow j{<=}i$
⟨*proof*⟩

lemma *Ord-trans*: $[\![i{\in}j;\ \ j{\in}k;\ \ Ord(k)]\!] \Longrightarrow i{\in}k$
⟨*proof*⟩

lemma *Ord-succ-subsetI*: $[\![i{\in}j;\ \ Ord(j)]\!] \Longrightarrow succ(i) \subseteq j$
⟨*proof*⟩

13.3 The Construction of Ordinals: 0, succ, Union

lemma *Ord-0* [*iff*,*TC*]: $Ord(0)$
⟨*proof*⟩

lemma *Ord-succ* [*TC*]: $Ord(i) \Longrightarrow Ord(succ(i))$
⟨*proof*⟩

lemmas *Ord-1* = *Ord-0* [*THEN Ord-succ*]

lemma *Ord-succ-iff* [*iff*]: $Ord(succ(i)) <-> Ord(i)$
⟨*proof*⟩

lemma *Ord-Un* [*intro*,*simp*,*TC*]: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow Ord(i \cup j)$
⟨*proof*⟩

lemma *Ord-Int* [*TC*]: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow Ord(i \cap j)$
⟨*proof*⟩

There is no set of all ordinals, for then it would contain itself

lemma *ON-class*: $\neg\ (\forall\, i.\ i{\in}X <-> Ord(i))$
⟨*proof*⟩

13.4 < is 'less Than' for Ordinals

lemma *ltI*: $[\![i{\in}j;\ \ Ord(j)]\!] \Longrightarrow i{<}j$
⟨*proof*⟩

lemma *ltE*:
$[\![i{<}j;\ [\![i{\in}j;\ \ Ord(i);\ \ Ord(j)]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *ltD*: $i<j \Longrightarrow i\in j$
⟨*proof*⟩

lemma *not-lt0* [*simp*]: $\neg\ i<0$
⟨*proof*⟩

lemma *lt-Ord*: $j<i \Longrightarrow Ord(j)$
⟨*proof*⟩

lemma *lt-Ord2*: $j<i \Longrightarrow Ord(i)$
⟨*proof*⟩

lemmas *le-Ord2* = *lt-Ord2* [*THEN Ord-succD*]

lemmas *lt0E* = *not-lt0* [*THEN notE*, *elim*!]

lemma *lt-trans* [*trans*]: $[\![i<j;\ \ j<k]\!] \Longrightarrow i<k$
⟨*proof*⟩

lemma *lt-not-sym*: $i<j \Longrightarrow \neg\ (j<i)$
⟨*proof*⟩

lemmas *lt-asym* = *lt-not-sym* [*THEN swap*]

lemma *lt-irrefl* [*elim*!]: $i<i \Longrightarrow P$
⟨*proof*⟩

lemma *lt-not-refl*: $\neg\ i<i$
⟨*proof*⟩

Recall that $i \le j$ abbreviates $i \le j$!

lemma *le-iff*: $i \le j <-> i<j \mid (i=j \wedge Ord(j))$
⟨*proof*⟩

lemma *leI*: $i<j \Longrightarrow i \le j$
⟨*proof*⟩

lemma *le-eqI*: $[\![i=j;\ \ Ord(j)]\!] \Longrightarrow i \le j$
⟨*proof*⟩

lemmas *le-refl* = *refl* [*THEN le-eqI*]

lemma *le-refl-iff* [*iff*]: $i \le i <-> Ord(i)$
⟨*proof*⟩

lemma *leCI*: $(\neg\ (i{=}j \wedge Ord(j)) \Longrightarrow i{<}j) \Longrightarrow i \leq j$
⟨*proof*⟩

lemma *leE*:
$[\![i \leq j;\ \ i{<}j \Longrightarrow P;\ \ [\![i{=}j;\ \ Ord(j)]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *le-anti-sym*: $[\![i \leq j;\ \ j \leq i]\!] \Longrightarrow i{=}j$
⟨*proof*⟩

lemma *le0-iff* [*simp*]: $i \leq 0 \ <\!\!-\!\!>\ i{=}0$
⟨*proof*⟩

lemmas *le0D* = *le0-iff* [*THEN iffD1*, *dest*!]

13.5 Natural Deduction Rules for Memrel

lemma *Memrel-iff* [*simp*]: $\langle a,b\rangle \in Memrel(A) \ <\!\!-\!\!>\ a{\in}b \wedge a{\in}A \wedge b{\in}A$
⟨*proof*⟩

lemma *MemrelI* [*intro*!]: $[\![a \in b;\ \ a \in A;\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle \in Memrel(A)$
⟨*proof*⟩

lemma *MemrelE* [*elim*!]:
$[\![\langle a,b\rangle \in Memrel(A);$
$\quad [\![a \in A;\ \ b \in A;\ \ a{\in}b]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
⟨*proof*⟩

lemma *Memrel-type*: $Memrel(A) \subseteq A{*}A$
⟨*proof*⟩

lemma *Memrel-mono*: $A{<=}B \Longrightarrow Memrel(A) \subseteq Memrel(B)$
⟨*proof*⟩

lemma *Memrel-0* [*simp*]: $Memrel(0) = 0$
⟨*proof*⟩

lemma *Memrel-1* [*simp*]: $Memrel(1) = 0$
⟨*proof*⟩

lemma *relation-Memrel*: $relation(Memrel(A))$
⟨*proof*⟩

lemma *wf-Memrel*: $wf(Memrel(A))$
⟨*proof*⟩

The premise $Ord(i)$ does not suffice.

lemma *trans-Memrel*:
$Ord(i) \Longrightarrow trans(Memrel(i))$
⟨*proof*⟩

However, the following premise is strong enough.

lemma *Transset-trans-Memrel*:
$\forall j{\in}i.\ Transset(j) \Longrightarrow trans(Memrel(i))$
⟨*proof*⟩

lemma *Transset-Memrel-iff*:
$Transset(A) \Longrightarrow \langle a,b\rangle \in Memrel(A) <-> a{\in}b \wedge b{\in}A$
⟨*proof*⟩

13.6 Transfinite Induction

lemma *Transset-induct*:
$[\![i \in k;\ \ Transset(k);$
$\bigwedge x.[\![x \in k;\ \forall y{\in}x.\ P(y)]\!] \Longrightarrow P(x)]\!]$
$\Longrightarrow\ P(i)$
⟨*proof*⟩

lemma *Ord-induct* [*consumes 2*]:
$i \in k \Longrightarrow Ord(k) \Longrightarrow (\bigwedge x.\ x \in k \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$
⟨*proof*⟩

lemma *trans-induct* [*consumes 1*, *case-names step*]:
$Ord(i) \Longrightarrow (\bigwedge x.\ Ord(x) \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$
⟨*proof*⟩

14 Fundamental properties of the epsilon ordering (< on ordinals)

14.0.1 Proving That < is a Linear Ordering on the Ordinals

lemma *Ord-linear*:
$Ord(i) \Longrightarrow Ord(j) \Longrightarrow i{\in}j \mid i{=}j \mid j{\in}i$
⟨*proof*⟩

The trichotomy law for ordinals

lemma *Ord-linear-lt*:
assumes *o*: $Ord(i)$ $Ord(j)$
obtains (*lt*) $i{<}j$ | (*eq*) $i{=}j$ | (*gt*) $j{<}i$
⟨*proof*⟩

lemma *Ord-linear2*:

assumes *o*: *Ord*(*i*) *Ord*(*j*)
obtains (*lt*) *i*<*j* | (*ge*) *j* ≤ *i*
⟨*proof*⟩

lemma *Ord-linear-le*:
assumes *o*: *Ord*(*i*) *Ord*(*j*)
obtains (*le*) *i* ≤ *j* | (*ge*) *j* ≤ *i*
⟨*proof*⟩

lemma *le-imp-not-lt*: *j* ≤ *i* ⟹ ¬ *i*<*j*
⟨*proof*⟩

lemma *not-lt-imp-le*: ⟦¬ *i*<*j*; *Ord*(*i*); *Ord*(*j*)⟧ ⟹ *j* ≤ *i*
⟨*proof*⟩

14.0.2 Some Rewrite Rules for <, ≤

lemma *Ord-mem-iff-lt*: *Ord*(*j*) ⟹ *i*∈*j* <−> *i*<*j*
⟨*proof*⟩

lemma *not-lt-iff-le*: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ ¬ *i*<*j* <−> *j* ≤ *i*
⟨*proof*⟩

lemma *not-le-iff-lt*: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ ¬ *i* ≤ *j* <−> *j*<*i*
⟨*proof*⟩

lemma *Ord-0-le*: *Ord*(*i*) ⟹ 0 ≤ *i*
⟨*proof*⟩

lemma *Ord-0-lt*: ⟦*Ord*(*i*); *i*≠0⟧ ⟹ 0<*i*
⟨*proof*⟩

lemma *Ord-0-lt-iff*: *Ord*(*i*) ⟹ *i*≠0 <−> 0<*i*
⟨*proof*⟩

14.1 Results about Less-Than or Equals

lemma *zero-le-succ-iff* [*iff*]: 0 ≤ *succ*(*x*) <−> *Ord*(*x*)
⟨*proof*⟩

lemma *subset-imp-le*: ⟦*j*<=*i*; *Ord*(*i*); *Ord*(*j*)⟧ ⟹ *j* ≤ *i*
⟨*proof*⟩

lemma *le-imp-subset*: *i* ≤ *j* ⟹ *i*<=*j*
⟨*proof*⟩

lemma *le-subset-iff*: *j* ≤ *i* <−> *j*<=*i* ∧ *Ord*(*i*) ∧ *Ord*(*j*)
⟨*proof*⟩

lemma *le-succ-iff*: $i \leq succ(j) <-> i \leq j \mid i{=}succ(j) \wedge Ord(i)$
⟨*proof*⟩

lemma *all-lt-imp-le*: $[\![Ord(i);\ \ Ord(j);\ \ \bigwedge x.\ x{<}j \Longrightarrow x{<}i]\!] \Longrightarrow j \leq i$
⟨*proof*⟩

14.1.1 Transitivity Laws

lemma *lt-trans1*: $[\![i \leq j;\ \ j{<}k]\!] \Longrightarrow i{<}k$
⟨*proof*⟩

lemma *lt-trans2*: $[\![i{<}j;\ \ j \leq k]\!] \Longrightarrow i{<}k$
⟨*proof*⟩

lemma *le-trans*: $[\![i \leq j;\ \ j \leq k]\!] \Longrightarrow i \leq k$
⟨*proof*⟩

lemma *succ-leI*: $i{<}j \Longrightarrow succ(i) \leq j$
⟨*proof*⟩

lemma *succ-leE*: $succ(i) \leq j \Longrightarrow i{<}j$
⟨*proof*⟩

lemma *succ-le-iff* [*iff*]: $succ(i) \leq j <-> i{<}j$
⟨*proof*⟩

lemma *succ-le-imp-le*: $succ(i) \leq succ(j) \Longrightarrow i \leq j$
⟨*proof*⟩

lemma *lt-subset-trans*: $[\![i \subseteq j;\ \ j{<}k;\ \ Ord(i)]\!] \Longrightarrow i{<}k$
⟨*proof*⟩

lemma *lt-imp-0-lt*: $j{<}i \Longrightarrow 0{<}i$
⟨*proof*⟩

lemma *succ-lt-iff*: $succ(i) < j <-> i{<}j \wedge succ(i) \neq j$
⟨*proof*⟩

lemma *Ord-succ-mem-iff*: $Ord(j) \Longrightarrow succ(i) \in succ(j) <-> i{\in}j$
⟨*proof*⟩

14.1.2 Union and Intersection

lemma *Un-upper1-le*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \leq i \cup j$
⟨*proof*⟩

lemma *Un-upper2-le*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow j \leq i \cup j$
⟨*proof*⟩

lemma *Un-least-lt*: $[\![i<k;\ \ j<k]\!] \Longrightarrow i \cup j < k$
⟨*proof*⟩

lemma *Un-least-lt-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \cup j < k\ \ <\!\!-\!\!>\ \ i<k \wedge j<k$
⟨*proof*⟩

lemma *Un-least-mem-iff*:
$[\![Ord(i);\ Ord(j);\ Ord(k)]\!] \Longrightarrow i \cup j \in k\ \ <\!\!-\!\!>\ \ i\in k \wedge j\in k$
⟨*proof*⟩

lemma *Int-greatest-lt*: $[\![i<k;\ \ j<k]\!] \Longrightarrow i \cap j < k$
⟨*proof*⟩

lemma *Ord-Un-if*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow i \cup j = (\mathit{if}\ j<i\ \mathit{then}\ i\ \mathit{else}\ j)$
⟨*proof*⟩

lemma *succ-Un-distrib*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow succ(i \cup j) = succ(i) \cup succ(j)$
⟨*proof*⟩

lemma *lt-Un-iff*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow k < i \cup j\ <\!\!-\!\!>\ k < i \mid k < j$
⟨*proof*⟩

lemma *le-Un-iff*:
$[\![Ord(i);\ Ord(j)]\!] \Longrightarrow k \le i \cup j\ <\!\!-\!\!>\ k \le i \mid k \le j$
⟨*proof*⟩

lemma *Un-upper1-lt*: $[\![k < i;\ Ord(j)]\!] \Longrightarrow k < i \cup j$
⟨*proof*⟩

lemma *Un-upper2-lt*: $[\![k < j;\ Ord(i)]\!] \Longrightarrow k < i \cup j$
⟨*proof*⟩

lemma *Ord-Union-succ-eq*: $Ord(i) \Longrightarrow \bigcup(succ(i)) = i$
⟨*proof*⟩

14.2 Results about Limits

lemma *Ord-Union* [*intro*,*simp*,*TC*]: $[\![\bigwedge i.\ i\in A \Longrightarrow Ord(i)]\!] \Longrightarrow Ord(\bigcup(A))$
⟨*proof*⟩

lemma *Ord-UN* [*intro*,*simp*,*TC*]:
$[\![\bigwedge x.\ x\in A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcup x\in A.\ B(x))$

⟨*proof*⟩

lemma *Ord-Inter* [*intro*,*simp*,*TC*]:
$[\![\bigwedge i.\ i{\in}A \Longrightarrow Ord(i)]\!] \Longrightarrow Ord(\bigcap(A))$
⟨*proof*⟩

lemma *Ord-INT* [*intro*,*simp*,*TC*]:
$[\![\bigwedge x.\ x{\in}A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcap x{\in}A.\ B(x))$
⟨*proof*⟩

lemma *UN-least-le*:
$[\![Ord(i);\ \bigwedge x.\ x{\in}A \Longrightarrow b(x) \le i]\!] \Longrightarrow (\bigcup x{\in}A.\ b(x)) \le i$
⟨*proof*⟩

lemma *UN-succ-least-lt*:
$[\![j{<}i;\ \bigwedge x.\ x{\in}A \Longrightarrow b(x){<}j]\!] \Longrightarrow (\bigcup x{\in}A.\ succ(b(x))) < i$
⟨*proof*⟩

lemma *UN-upper-lt*:
$[\![a{\in}A;\ i < b(a);\ Ord(\bigcup x{\in}A.\ b(x))]\!] \Longrightarrow i < (\bigcup x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *UN-upper-le*:
$[\![a \in A;\ i \le b(a);\ Ord(\bigcup x{\in}A.\ b(x))]\!] \Longrightarrow i \le (\bigcup x{\in}A.\ b(x))$
⟨*proof*⟩

lemma *lt-Union-iff*: $\forall\, i{\in}A.\ Ord(i) \Longrightarrow (j < \bigcup(A)) <\!\!-\!\!> (\exists\, i{\in}A.\ j{<}i)$
⟨*proof*⟩

lemma *Union-upper-le*:
$[\![j \in J;\ i{\le}j;\ Ord(\bigcup(J))]\!] \Longrightarrow i \le \bigcup J$
⟨*proof*⟩

lemma *le-implies-UN-le-UN*:
$[\![\bigwedge x.\ x{\in}A \Longrightarrow c(x) \le d(x)]\!] \Longrightarrow (\bigcup x{\in}A.\ c(x)) \le (\bigcup x{\in}A.\ d(x))$
⟨*proof*⟩

lemma *Ord-equality*: $Ord(i) \Longrightarrow (\bigcup y{\in}i.\ succ(y)) = i$
⟨*proof*⟩

lemma *Ord-Union-subset*: $Ord(i) \Longrightarrow \bigcup(i) \subseteq i$
⟨*proof*⟩

14.3 Limit Ordinals – General Properties

lemma *Limit-Union-eq*: $Limit(i) \Longrightarrow \bigcup(i) = i$

⟨*proof*⟩

lemma *Limit-is-Ord*: $Limit(i) \Longrightarrow Ord(i)$
⟨*proof*⟩

lemma *Limit-has-0*: $Limit(i) \Longrightarrow 0 < i$
⟨*proof*⟩

lemma *Limit-nonzero*: $Limit(i) \Longrightarrow i \neq 0$
⟨*proof*⟩

lemma *Limit-has-succ*: $[\![Limit(i);\ j<i]\!] \Longrightarrow succ(j) < i$
⟨*proof*⟩

lemma *Limit-succ-lt-iff* [*simp*]: $Limit(i) \Longrightarrow succ(j) < i <-> (j<i)$
⟨*proof*⟩

lemma *zero-not-Limit* [*iff*]: $\neg\ Limit(0)$
⟨*proof*⟩

lemma *Limit-has-1*: $Limit(i) \Longrightarrow 1 < i$
⟨*proof*⟩

lemma *increasing-LimitI*: $[\![0<l;\ \forall x{\in}l.\ \exists y{\in}l.\ x<y]\!] \Longrightarrow Limit(l)$
⟨*proof*⟩

lemma *non-succ-LimitI*:
assumes *i*: $0<i$ **and** *nsucc*: $\bigwedge y.\ succ(y) \neq i$
shows $Limit(i)$
⟨*proof*⟩

lemma *succ-LimitE* [*elim*!]: $Limit(succ(i)) \Longrightarrow P$
⟨*proof*⟩

lemma *not-succ-Limit* [*simp*]: $\neg\ Limit(succ(i))$
⟨*proof*⟩

lemma *Limit-le-succD*: $[\![Limit(i);\ i \leq succ(j)]\!] \Longrightarrow i \leq j$
⟨*proof*⟩

14.3.1 Traditional 3-Way Case Analysis on Ordinals

lemma *Ord-cases-disj*: $Ord(i) \Longrightarrow i=0 \mid (\exists j.\ Ord(j) \wedge i=succ(j)) \mid Limit(i)$
⟨*proof*⟩

lemma *Ord-cases*:
assumes *i*: $Ord(i)$
obtains (*0*) $i=0$ | (*succ*) *j* **where** $Ord(j)$ $i=succ(j)$ | (*limit*) $Limit(i)$
⟨*proof*⟩

lemma *trans-induct3-raw*:
$$\llbracket Ord(i);\ P(0);\ \bigwedge x.\ \llbracket Ord(x);\ P(x)\rrbracket \Longrightarrow P(succ(x));\ \bigwedge x.\ \llbracket Limit(x);\ \forall y{\in}x.\ P(y)\rrbracket \Longrightarrow P(x) \rrbracket \Longrightarrow P(i)$$
⟨*proof*⟩

lemma *trans-induct3* [*case-names 0 succ limit*, *consumes 1*]:
$$Ord(i) \Longrightarrow P(0) \Longrightarrow (\bigwedge x.\ Ord(x) \Longrightarrow P(x) \Longrightarrow P(succ(x))) \Longrightarrow (\bigwedge x.\ Limit(x) \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$$
⟨*proof*⟩

A set of ordinals is either empty, contains its own union, or its union is a limit ordinal.

lemma *Union-le*: $\llbracket \bigwedge x.\ x{\in}I \Longrightarrow x{\le}j;\ Ord(j)\rrbracket \Longrightarrow \bigcup(I) \le j$
⟨*proof*⟩

lemma *Ord-set-cases*:
assumes I: $\forall\, i{\in}I.\ Ord(i)$
shows $I{=}0 \vee \bigcup(I) \in I \vee (\bigcup(I) \notin I \wedge Limit(\bigcup(I)))$
⟨*proof*⟩

If the union of a set of ordinals is a successor, then it is an element of that set.

lemma *Ord-Union-eq-succD*: $\llbracket \forall\, x{\in}X.\ Ord(x);\ \bigcup X = succ(j)\rrbracket \Longrightarrow succ(j) \in X$
⟨*proof*⟩

lemma *Limit-Union* [*rule-format*]: $\llbracket I \neq 0;\ (\bigwedge i.\ i{\in}I \Longrightarrow Limit(i))\rrbracket \Longrightarrow Limit(\bigcup I)$
⟨*proof*⟩

end

15 Special quantifiers

theory *OrdQuant* **imports** *Ordinal* **begin**

15.1 Quantifiers and union operator for ordinals

definition

$oall :: [i,\ i \Rightarrow o] \Rightarrow o$ **where**
$oall(A,\ P) \equiv \forall\, x.\ x{<}A \longrightarrow P(x)$

definition
$oex :: [i,\ i \Rightarrow o] \Rightarrow o$ **where**
$oex(A,\ P) \equiv \exists\, x.\ x{<}A \wedge P(x)$

definition

$OUnion :: [i,\ i \Rightarrow i] \Rightarrow i$ **where**
$OUnion(i,B) \equiv \{z{:}\ \bigcup x{\in}i.\ B(x).\ Ord(i)\}$

syntax
$\text{-}oall :: [idt,\ i,\ o] \Rightarrow o$ $(\langle(3\forall\text{-}{<}\text{-}./\ \text{-})\rangle\ 10)$
$\text{-}oex :: [idt,\ i,\ o] \Rightarrow o$ $(\langle(3\exists\text{-}{<}\text{-}./\ \text{-})\rangle\ 10)$
$\text{-}OUNION :: [idt,\ i,\ i] \Rightarrow i$ $(\langle(3\bigcup\text{-}{<}\text{-}./\ \text{-})\rangle\ 10)$
translations
$\forall x{<}a.\ P \rightleftharpoons CONST\ oall(a,\ \lambda x.\ P)$
$\exists x{<}a.\ P \rightleftharpoons CONST\ oex(a,\ \lambda x.\ P)$
$\bigcup x{<}a.\ B \rightleftharpoons CONST\ OUnion(a,\ \lambda x.\ B)$

15.1.1 simplification of the new quantifiers

lemma [*simp*]: $(\forall x{<}0.\ P(x))$
⟨*proof*⟩

lemma [*simp*]: $\neg(\exists x{<}0.\ P(x))$
⟨*proof*⟩

lemma [*simp*]: $(\forall x{<}succ(i).\ P(x)) <\!\!-\!\!> (Ord(i) \longrightarrow P(i) \wedge (\forall x{<}i.\ P(x)))$
⟨*proof*⟩

lemma [*simp*]: $(\exists x{<}succ(i).\ P(x)) <\!\!-\!\!> (Ord(i) \wedge (P(i) \mid (\exists x{<}i.\ P(x))))$
⟨*proof*⟩

15.1.2 Union over ordinals

lemma *Ord-OUN* [*intro*,*simp*]:
$[\![\bigwedge x.\ x{<}A \Longrightarrow Ord(B(x))]\!] \Longrightarrow Ord(\bigcup x{<}A.\ B(x))$
⟨*proof*⟩

lemma *OUN-upper-lt*:
$[\![a{<}A;\ \ i < b(a);\ \ Ord(\bigcup x{<}A.\ b(x))]\!] \Longrightarrow i < (\bigcup x{<}A.\ b(x))$
⟨*proof*⟩

lemma *OUN-upper-le*:
$[\![a{<}A;\ \ i{\le}b(a);\ \ Ord(\bigcup x{<}A.\ b(x))]\!] \Longrightarrow i \le (\bigcup x{<}A.\ b(x))$
⟨*proof*⟩

lemma *Limit-OUN-eq*: $Limit(i) \Longrightarrow (\bigcup x{<}i.\ x) = i$
⟨*proof*⟩

lemma *OUN-least*:
$(\bigwedge x.\ x{<}A \Longrightarrow B(x) \subseteq C) \Longrightarrow (\bigcup x{<}A.\ B(x)) \subseteq C$
⟨*proof*⟩

lemma *OUN-least-le*:
$[\![Ord(i);\ \bigwedge x.\ x<A \Longrightarrow b(x) \le i]\!] \Longrightarrow (\bigcup x<A.\ b(x)) \le i$
⟨*proof*⟩

lemma *le-implies-OUN-le-OUN*:
$[\![\bigwedge x.\ x<A \Longrightarrow c(x) \le d(x)]\!] \Longrightarrow (\bigcup x<A.\ c(x)) \le (\bigcup x<A.\ d(x))$
⟨*proof*⟩

lemma *OUN-UN-eq*:
$(\bigwedge x.\ x \in A \Longrightarrow Ord(B(x)))$
$\Longrightarrow (\bigcup z < (\bigcup x \in A.\ B(x)).\ C(z)) = (\bigcup x \in A.\ \bigcup z < B(x).\ C(z))$
⟨*proof*⟩

lemma *OUN-Union-eq*:
$(\bigwedge x.\ x \in X \Longrightarrow Ord(x))$
$\Longrightarrow (\bigcup z < \bigcup(X).\ C(z)) = (\bigcup x \in X.\ \bigcup z < x.\ C(z))$
⟨*proof*⟩

lemma *atomize-oall* [*symmetric*, *rulify*]:
$(\bigwedge x.\ x<A \Longrightarrow P(x)) \equiv Trueprop\ (\forall x<A.\ P(x))$
⟨*proof*⟩

15.1.3 universal quantifier for ordinals

lemma *oallI* [*intro*!]:
$[\![\bigwedge x.\ x<A \Longrightarrow P(x)]\!] \Longrightarrow \forall x<A.\ P(x)$
⟨*proof*⟩

lemma *ospec*: $[\![\forall x<A.\ P(x);\ x<A]\!] \Longrightarrow P(x)$
⟨*proof*⟩

lemma *oallE*:
$[\![\forall x<A.\ P(x);\ P(x) \Longrightarrow Q;\ \neg x<A \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *rev-oallE* [*elim*]:
$[\![\forall x<A.\ P(x);\ \neg x<A \Longrightarrow Q;\ P(x) \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *oall-simp* [*simp*]: $(\forall x<a.\ True) <-> True$
⟨*proof*⟩

lemma *oall-cong* [*cong*]:
$[\![a=a';\ \bigwedge x.\ x<a' \Longrightarrow P(x) <-> P'(x)]\!]$

$\Longrightarrow$ *oall*(*a*, $\lambda x.\ P(x)$) <-> *oall*(a', $\lambda x.\ P'(x)$)
⟨*proof*⟩

15.1.4 existential quantifier for ordinals

lemma *oexI* [*intro*]:
$[\![P(x);\ x<A]\!] \Longrightarrow \exists\, x<A.\ P(x)$
⟨*proof*⟩

lemma *oexCI*:
$[\![\forall\, x<A.\ \neg P(x) \Longrightarrow P(a);\ a<A]\!] \Longrightarrow \exists\, x<A.\ P(x)$
⟨*proof*⟩

lemma *oexE* [*elim*!]:
$[\![\exists\, x<A.\ P(x);\ \bigwedge x.\ [\![x<A;\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *oex-cong* [*cong*]:
$[\![a=a';\ \bigwedge x.\ x<a' \Longrightarrow P(x) <-> P'(x)]\!]$
$\Longrightarrow$ *oex*(*a*, $\lambda x.\ P(x)$) <-> *oex*(a', $\lambda x.\ P'(x)$)
⟨*proof*⟩

15.1.5 Rules for Ordinal-Indexed Unions

lemma *OUN-I* [*intro*]: $[\![a<i;\ b \in B(a)]\!] \Longrightarrow b$: $(\bigcup z<i.\ B(z))$
⟨*proof*⟩

lemma *OUN-E* [*elim*!]:
$[\![b \in (\bigcup z<i.\ B(z));\ \bigwedge a.[\![b \in B(a);\ a<i]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *OUN-iff*: $b \in (\bigcup x<i.\ B(x))$ <-> $(\exists\, x<i.\ b \in B(x))$
⟨*proof*⟩

lemma *OUN-cong* [*cong*]:
$[\![i=j;\ \bigwedge x.\ x<j \Longrightarrow C(x)=D(x)]\!] \Longrightarrow (\bigcup x<i.\ C(x)) = (\bigcup x<j.\ D(x))$
⟨*proof*⟩

lemma *lt-induct*:
$[\![i<k;\ \bigwedge x.[\![x<k;\ \forall\, y<x.\ P(y)]\!] \Longrightarrow P(x)]\!] \Longrightarrow P(i)$
⟨*proof*⟩

15.2 Quantification over a class

definition
rall :: $[i\Rightarrow o,\ i\Rightarrow o] \Rightarrow o$ **where**
rall(M, P) $\equiv \forall\, x.\ M(x) \longrightarrow P(x)$

definition

rex :: $[i \Rightarrow o,\ i \Rightarrow o] \Rightarrow o$ **where**
$rex(M,\ P) \equiv \exists\, x.\ M(x) \wedge P(x)$

syntax
-rall :: $[pttrn,\ i \Rightarrow o,\ o] \Rightarrow o$ ($\langle(3\forall$ -[-]./ -)$\rangle$ *10*)
-rex :: $[pttrn,\ i \Rightarrow o,\ o] \Rightarrow o$ ($\langle(3\exists$ -[-]./ -)$\rangle$ *10*)
translations
$\forall\, x[M].\ P \rightleftharpoons CONST\ rall(M,\ \lambda x.\ P)$
$\exists\, x[M].\ P \rightleftharpoons CONST\ rex(M,\ \lambda x.\ P)$

15.2.1 Relativized universal quantifier

lemma *rallI* [*intro*!]: $[\![\bigwedge x.\ M(x) \Longrightarrow P(x)]\!] \Longrightarrow \forall\, x[M].\ P(x)$
⟨*proof*⟩

lemma *rspec*: $[\![\forall\, x[M].\ P(x);\ M(x)]\!] \Longrightarrow P(x)$
⟨*proof*⟩

lemma *rev-rallE* [*elim*]:
$[\![\forall\, x[M].\ P(x);\ \neg\ M(x) \Longrightarrow Q;\ \ P(x) \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *rallE*: $[\![\forall\, x[M].\ P(x);\ \ P(x) \Longrightarrow Q;\ \neg\ M(x) \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *rall-triv* [*simp*]: $(\forall\, x[M].\ P) \longleftrightarrow ((\exists\, x.\ M(x)) \longrightarrow P)$
⟨*proof*⟩

lemma *rall-cong* [*cong*]:
$(\bigwedge x.\ M(x) \Longrightarrow P(x) <-> P'(x)) \Longrightarrow (\forall\, x[M].\ P(x)) <-> (\forall\, x[M].\ P'(x))$
⟨*proof*⟩

15.2.2 Relativized existential quantifier

lemma *rexI* [*intro*]: $[\![P(x);\ M(x)]\!] \Longrightarrow \exists\, x[M].\ P(x)$
⟨*proof*⟩

lemma *rev-rexI*: $[\![M(x);\ \ P(x)]\!] \Longrightarrow \exists\, x[M].\ P(x)$
⟨*proof*⟩

lemma *rexCI*: $[\![\forall\, x[M].\ \neg P(x) \Longrightarrow P(a);\ M(a)]\!] \Longrightarrow \exists\, x[M].\ P(x)$
⟨*proof*⟩

lemma *rexE* [*elim*!]: $[\![\exists\, x[M].\ P(x);\ \ \bigwedge x.\ [\![M(x);\ P(x)]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *rex-triv* [*simp*]: $(\exists\, x[M].\ P) \longleftrightarrow ((\exists\, x.\ M(x)) \wedge P)$
⟨*proof*⟩

lemma *rex-cong* [*cong*]:
$(\bigwedge x.\ M(x) \Longrightarrow P(x) <-> P'(x)) \Longrightarrow (\exists\, x[M].\ P(x)) <-> (\exists\, x[M].\ P'(x))$
⟨*proof*⟩

lemma *rall-is-ball* [*simp*]: $(\forall\, x[\lambda z.\ z{\in}A].\ P(x)) <-> (\forall\, x{\in}A.\ P(x))$
⟨*proof*⟩

lemma *rex-is-bex* [*simp*]: $(\exists\, x[\lambda z.\ z{\in}A].\ P(x)) <-> (\exists\, x{\in}A.\ P(x))$
⟨*proof*⟩

lemma *atomize-rall*: $(\bigwedge x.\ M(x) \Longrightarrow P(x)) \equiv \mathit{Trueprop}\ (\forall\, x[M].\ P(x))$
⟨*proof*⟩

declare *atomize-rall* [*symmetric*, *rulify*]

lemma *rall-simps1*:
$(\forall\, x[M].\ P(x) \wedge Q) \quad <-> (\forall\, x[M].\ P(x)) \wedge ((\forall\, x[M].\ \mathit{False}) \mid Q)$
$(\forall\, x[M].\ P(x) \mid Q) \quad <-> ((\forall\, x[M].\ P(x)) \mid Q)$
$(\forall\, x[M].\ P(x) \longrightarrow Q) <-> ((\exists\, x[M].\ P(x)) \longrightarrow Q)$
$(\neg(\forall\, x[M].\ P(x))) <-> (\exists\, x[M].\ \neg P(x))$
⟨*proof*⟩

lemma *rall-simps2*:
$(\forall\, x[M].\ P \wedge Q(x)) \quad <-> ((\forall\, x[M].\ \mathit{False}) \mid P) \wedge (\forall\, x[M].\ Q(x))$
$(\forall\, x[M].\ P \mid Q(x)) \quad <-> (P \mid (\forall\, x[M].\ Q(x)))$
$(\forall\, x[M].\ P \longrightarrow Q(x)) <-> (P \longrightarrow (\forall\, x[M].\ Q(x)))$
⟨*proof*⟩

lemmas *rall-simps* [*simp*] = *rall-simps1 rall-simps2*

lemma *rall-conj-distrib*:
$(\forall\, x[M].\ P(x) \wedge Q(x)) <-> ((\forall\, x[M].\ P(x)) \wedge (\forall\, x[M].\ Q(x)))$
⟨*proof*⟩

lemma *rex-simps1*:
$(\exists\, x[M].\ P(x) \wedge Q) <-> ((\exists\, x[M].\ P(x)) \wedge Q)$
$(\exists\, x[M].\ P(x) \mid Q) <-> (\exists\, x[M].\ P(x)) \mid ((\exists\, x[M].\ \mathit{True}) \wedge Q)$
$(\exists\, x[M].\ P(x) \longrightarrow Q) <-> ((\forall\, x[M].\ P(x)) \longrightarrow ((\exists\, x[M].\ \mathit{True}) \wedge Q))$
$(\neg(\exists\, x[M].\ P(x))) <-> (\forall\, x[M].\ \neg P(x))$
⟨*proof*⟩

lemma *rex-simps2*:
$(\exists\, x[M].\ P \wedge Q(x)) <-> (P \wedge (\exists\, x[M].\ Q(x)))$
$(\exists\, x[M].\ P \mid Q(x)) <-> ((\exists\, x[M].\ \mathit{True}) \wedge P) \mid (\exists\, x[M].\ Q(x))$

$(\exists\, x[M].\ P \longrightarrow Q(x)) <-> (((\forall\, x[M].\ \mathit{False}) \mid P) \longrightarrow (\exists\, x[M].\ Q(x)))$
⟨*proof*⟩

lemmas *rex-simps* [*simp*] = *rex-simps1 rex-simps2*

lemma *rex-disj-distrib*:
$(\exists\, x[M].\ P(x) \mid Q(x)) <-> ((\exists\, x[M].\ P(x)) \mid (\exists\, x[M].\ Q(x)))$
⟨*proof*⟩

15.2.3 One-point rule for bounded quantifiers

lemma *rex-triv-one-point1* [*simp*]: $(\exists\, x[M].\ x{=}a) <-> (\ M(a))$
⟨*proof*⟩

lemma *rex-triv-one-point2* [*simp*]: $(\exists\, x[M].\ a{=}x) <-> (\ M(a))$
⟨*proof*⟩

lemma *rex-one-point1* [*simp*]: $(\exists\, x[M].\ x{=}a \wedge P(x)) <-> (\ M(a) \wedge P(a))$
⟨*proof*⟩

lemma *rex-one-point2* [*simp*]: $(\exists\, x[M].\ a{=}x \wedge P(x)) <-> (\ M(a) \wedge P(a))$
⟨*proof*⟩

lemma *rall-one-point1* [*simp*]: $(\forall\, x[M].\ x{=}a \longrightarrow P(x)) <-> (\ M(a) \longrightarrow P(a))$
⟨*proof*⟩

lemma *rall-one-point2* [*simp*]: $(\forall\, x[M].\ a{=}x \longrightarrow P(x)) <-> (\ M(a) \longrightarrow P(a))$
⟨*proof*⟩

15.2.4 Sets as Classes

definition
setclass :: [*i*,*i*] ⇒ *o* (‹##-› [40] 40) **where**
setclass(*A*) ≡ λ*x*. *x* ∈ *A*

lemma *setclass-iff* [*simp*]: *setclass*(*A*,*x*) <-> *x* ∈ *A*
⟨*proof*⟩

lemma *rall-setclass-is-ball* [*simp*]: $(\forall\, x[\#\#A].\ P(x)) <-> (\forall\, x{\in}A.\ P(x))$
⟨*proof*⟩

lemma *rex-setclass-is-bex* [*simp*]: $(\exists\, x[\#\#A].\ P(x)) <-> (\exists\, x{\in}A.\ P(x))$
⟨*proof*⟩

⟨*ML*⟩

Setting up the one-point-rule simproc

⟨*ML*⟩

end

16 The Natural numbers As a Least Fixed Point

theory *Nat* **imports** *OrdQuant Bool* **begin**

definition
  nat :: *i* **where**
    $nat \equiv lfp(Inf, \lambda X.\ \{0\} \cup \{succ(i).\ i \in X\})$

definition
  quasinat :: $i \Rightarrow o$ **where**
    $quasinat(n) \equiv n=0 \mid (\exists\, m.\ n = succ(m))$

definition

  nat-case :: $[i,\ i \Rightarrow i,\ i] \Rightarrow i$ **where**
    $nat\text{-}case(a,b,k) \equiv THE\ y.\ k=0 \wedge y=a \mid (\exists\, x.\ k=succ(x) \wedge y=b(x))$

definition
  nat-rec :: $[i,\ i,\ [i,i] \Rightarrow i] \Rightarrow i$ **where**
    $nat\text{-}rec(k,a,b) \equiv$
        $wfrec(Memrel(nat),\ k,\ \lambda n\ f.\ nat\text{-}case(a,\ \lambda m.\ b(m,\ f`m),\ n))$

definition
  Le :: *i* **where**
    $Le \equiv \{\langle x,y\rangle : nat * nat.\ x \leq y\}$

definition
  Lt :: *i* **where**
    $Lt \equiv \{\langle x,\ y\rangle : nat * nat.\ x < y\}$

definition
  Ge :: *i* **where**
    $Ge \equiv \{\langle x,y\rangle : nat * nat.\ y \leq x\}$

definition
  Gt :: *i* **where**
    $Gt \equiv \{\langle x,y\rangle : nat * nat.\ y < x\}$

definition
  greater-than :: $i \Rightarrow i$ **where**
    $greater\text{-}than(n) \equiv \{i \in nat.\ n < i\}$

No need for a less-than operator: a natural number is its list of predecessors!

lemma *nat-bnd-mono*: $bnd\text{-}mono(Inf, \lambda X.\ \{0\} \cup \{succ(i).\ i \in X\})$
⟨*proof*⟩

lemmas *nat-unfold* = *nat-bnd-mono* [*THEN nat-def* [*THEN def-lfp-unfold*]]

lemma *nat-0I* [*iff*, *TC*]: $0 \in nat$
⟨*proof*⟩

lemma *nat-succI* [*intro*!, *TC*]: $n \in nat \Longrightarrow succ(n) \in nat$
⟨*proof*⟩

lemma *nat-1I* [*iff*, *TC*]: $1 \in nat$
⟨*proof*⟩

lemma *nat-2I* [*iff*, *TC*]: $2 \in nat$
⟨*proof*⟩

lemma *bool-subset-nat*: $bool \subseteq nat$
⟨*proof*⟩

lemmas *bool-into-nat* = *bool-subset-nat* [*THEN subsetD*]

16.1 Injectivity Properties and Induction

lemma *nat-induct* [*case-names 0 succ*, *induct set*: *nat*]:
$[\![n \in nat;\ P(0);\ \bigwedge x.\ [\![x \in nat;\ P(x)]\!] \Longrightarrow P(succ(x))]\!] \Longrightarrow P(n)$
⟨*proof*⟩

lemma *natE*:
assumes $n \in nat$
obtains (*0*) $n=0$ | (*succ*) x **where** $x \in nat$ $n=succ(x)$
⟨*proof*⟩

lemma *nat-into-Ord* [*simp*]: $n \in nat \Longrightarrow Ord(n)$
⟨*proof*⟩

lemmas *nat-0-le* = *nat-into-Ord* [*THEN Ord-0-le*]

lemmas *nat-le-refl* = *nat-into-Ord* [*THEN le-refl*]

lemma *Ord-nat* [*iff*]: $Ord(nat)$
⟨*proof*⟩

lemma *Limit-nat* [*iff*]: $Limit(nat)$
⟨*proof*⟩

lemma *naturals-not-limit*: $a \in nat \Longrightarrow \neg\ Limit(a)$
⟨*proof*⟩

lemma *succ-natD*: *succ*(*i*): $nat \Longrightarrow i \in nat$
⟨*proof*⟩

lemma *nat-succ-iff* [*iff*]: *succ*(*n*): $nat \longleftrightarrow n \in nat$
⟨*proof*⟩

lemma *nat-le-Limit*: $Limit(i) \Longrightarrow nat \leq i$
⟨*proof*⟩

lemmas *succ-in-naturalD* = *Ord-trans* [*OF succI1 - nat-into-Ord*]

lemma *lt-nat-in-nat*: $[\![m<n;\ \ n \in nat]\!] \Longrightarrow m \in nat$
⟨*proof*⟩

lemma *le-in-nat*: $[\![m \leq n;\ n \in nat]\!] \Longrightarrow m \in nat$
⟨*proof*⟩

16.2 Variations on Mathematical Induction

lemmas *complete-induct* = *Ord-induct* [*OF - Ord-nat*, *case-names less*, *consumes 1*]

lemma *complete-induct-rule* [*case-names less*, *consumes 1*]:
$i \in nat \Longrightarrow (\bigwedge x.\ x \in nat \Longrightarrow (\bigwedge y.\ y \in x \Longrightarrow P(y)) \Longrightarrow P(x)) \Longrightarrow P(i)$
⟨*proof*⟩

lemma *nat-induct-from*:
assumes $m \leq n\ m \in nat\ n \in nat$
and $P(m)$
and $\bigwedge x.\ [\![x \in nat;\ \ m \leq x;\ \ P(x)]\!] \Longrightarrow P(succ(x))$
shows $P(n)$
⟨*proof*⟩

lemma *diff-induct* [*case-names 0 0-succ succ-succ*, *consumes 2*]:
$[\![m \in nat;\ \ n \in nat;$
$\bigwedge x.\ x \in nat \Longrightarrow P(x,0);$
$\bigwedge y.\ y \in nat \Longrightarrow P(0,succ(y));$
$\bigwedge x\ y.\ [\![x \in nat;\ \ y \in nat;\ \ P(x,y)]\!] \Longrightarrow P(succ(x),succ(y))]\!]$
$\Longrightarrow P(m,n)$
⟨*proof*⟩

lemma *succ-lt-induct-lemma* [*rule-format*]:
$m \in nat \Longrightarrow P(m,succ(m)) \longrightarrow (\forall\, x{\in}nat.\ P(m,x) \longrightarrow P(m,succ(x))) \longrightarrow$
$(\forall\, n{\in}nat.\ m{<}n \longrightarrow P(m,n))$
⟨*proof*⟩

lemma *succ-lt-induct*:
$[\![m{<}n;\ \ n \in nat;$
$P(m,succ(m));$
$\bigwedge x.\ [\![x \in nat;\ \ P(m,x)]\!] \Longrightarrow P(m,succ(x))]\!]$
$\Longrightarrow P(m,n)$
⟨*proof*⟩

16.3 quasinat: to allow a case-split rule for *nat-case*

True if the argument is zero or any successor

lemma [*iff*]: *quasinat*(*0*)
⟨*proof*⟩

lemma [*iff*]: *quasinat*(*succ*(*x*))
⟨*proof*⟩

lemma *nat-imp-quasinat*: $n \in nat \Longrightarrow quasinat(n)$
⟨*proof*⟩

lemma *non-nat-case*: $\neg\ quasinat(x) \Longrightarrow nat\text{-}case(a,b,x) = 0$
⟨*proof*⟩

lemma *nat-cases-disj*: $k{=}0 \mid (\exists\, y.\ k = succ(y)) \mid \neg\ quasinat(k)$
⟨*proof*⟩

lemma *nat-cases*:
$[\![k{=}0 \Longrightarrow P;\ \ \bigwedge y.\ k = succ(y) \Longrightarrow P;\ \neg\ quasinat(k) \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *nat-case-0* [*simp*]: $nat\text{-}case(a,b,0) = a$
⟨*proof*⟩

lemma *nat-case-succ* [*simp*]: $nat\text{-}case(a,b,succ(n)) = b(n)$
⟨*proof*⟩

lemma *nat-case-type* [*TC*]:
$[\![n \in nat;\ \ a \in C(0);\ \ \bigwedge m.\ m \in nat \Longrightarrow b(m)\colon C(succ(m))]\!]$
$\Longrightarrow nat\text{-}case(a,b,n) \in C(n)$
⟨*proof*⟩

lemma *split-nat-case*:

$P(nat\text{-}case(a,b,k)) \longleftrightarrow$
$((k=0 \longrightarrow P(a)) \wedge (\forall x.\ k=succ(x) \longrightarrow P(b(x))) \wedge (\neg\ quasinat(k) \longrightarrow P(0)))$
⟨*proof*⟩

16.4 Recursion on the Natural Numbers

lemma *nat-rec-0*: $nat\text{-}rec(0,a,b) = a$
⟨*proof*⟩

lemma *nat-rec-succ*: $m \in nat \Longrightarrow nat\text{-}rec(succ(m),a,b) = b(m,\ nat\text{-}rec(m,a,b))$
⟨*proof*⟩

lemma *Un-nat-type* [*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow i \cup j \in nat$
⟨*proof*⟩

lemma *Int-nat-type* [*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow i \cap j \in nat$
⟨*proof*⟩

lemma *nat-nonempty* [*simp*]: $nat \neq 0$
⟨*proof*⟩

A natural number is the set of its predecessors

lemma *nat-eq-Collect-lt*: $i \in nat \Longrightarrow \{j{\in}nat.\ j{<}i\} = i$
⟨*proof*⟩

lemma *Le-iff* [*iff*]: $\langle x,y\rangle \in Le \longleftrightarrow x \leq y \wedge x \in nat \wedge y \in nat$
⟨*proof*⟩

end

17 Inductive and Coinductive Definitions

theory *Inductive*
imports *Fixedpt QPair Nat*
keywords
inductive coinductive inductive-cases rep-datatype primrec :: *thy-decl* **and**
domains intros monos con-defs type-intros type-elims
elimination induction case-eqns recursor-eqns :: *quasi-command*
begin

lemma *def-swap-iff*: $a \equiv b \Longrightarrow a = c \longleftrightarrow c = b$
⟨*proof*⟩

lemma *def-trans*: $f \equiv g \Longrightarrow g(a) = b \Longrightarrow f(a) = b$
⟨*proof*⟩

lemma *refl-thin*: $\bigwedge P.\ a = a \Longrightarrow P \Longrightarrow P$ ⟨*proof*⟩

⟨*ML*⟩

end

18 Epsilon Induction and Recursion

theory *Epsilon* **imports** *Nat* **begin**

definition
  eclose :: $i \Rightarrow i$ **where**
    $eclose(A) \equiv \bigcup n{\in}nat.\ nat\text{-}rec(n,\ A,\ \lambda m\ r.\ \bigcup(r))$

definition
  transrec :: $[i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
    $transrec(a,H) \equiv wfrec(Memrel(eclose(\{a\})),\ a,\ H)$

definition
  rank :: $i \Rightarrow i$ **where**
    $rank(a) \equiv transrec(a,\ \lambda x\ f.\ \bigcup y{\in}x.\ succ(f`y))$

definition
  transrec2 :: $[i,\ i,\ [i,i]\Rightarrow i] \Rightarrow i$ **where**
    $transrec2(k,\ a,\ b) \equiv$
      $transrec(k,$
        $\lambda i\ r.\ if(i{=}0,\ a,$
          $if(\exists j.\ i{=}succ(j),$
            $b(THE\ j.\ i{=}succ(j),\ r`(THE\ j.\ i{=}succ(j))),$
            $\bigcup j{<}i.\ r`j)))$

definition
  recursor :: $[i,\ [i,i]\Rightarrow i,\ i]\Rightarrow i$ **where**
    $recursor(a,b,k) \equiv\ transrec(k,\ \lambda n\ f.\ nat\text{-}case(a,\ \lambda m.\ b(m,\ f`m),\ n))$

definition
  rec :: $[i,\ i,\ [i,i]\Rightarrow i]\Rightarrow i$ **where**
    $rec(k,a,b) \equiv recursor(a,b,k)$

18.1 Basic Closure Properties

lemma *arg-subset-eclose*: $A \subseteq eclose(A)$
  ⟨*proof*⟩

lemmas *arg-into-eclose* = *arg-subset-eclose* [*THEN subsetD*]

lemma *Transset-eclose*: *Transset*$(eclose(A))$
  ⟨*proof*⟩

lemmas *eclose-subset* =
Transset-eclose [*unfolded Transset-def*, *THEN bspec*]

lemmas *ecloseD* = *eclose-subset* [*THEN subsetD*]

lemmas *arg-in-eclose-sing* = *arg-subset-eclose* [*THEN singleton-subsetD*]
lemmas *arg-into-eclose-sing* = *arg-in-eclose-sing* [*THEN ecloseD*]

lemmas *eclose-induct* =
Transset-induct [*OF - Transset-eclose*, *induct set*: *eclose*]

lemma *eps-induct*:
$[\![\bigwedge x.\ \forall\, y{\in}x.\ P(y) \Longrightarrow P(x)]\!] \Longrightarrow P(a)$
⟨*proof*⟩

18.2 Leastness of *eclose*

lemma *eclose-least-lemma*:
$[\![Transset(X);\ A{<=}X;\ n \in nat]\!] \Longrightarrow nat\text{-}rec(n, A, \lambda m\ r.\ \bigcup(r)) \subseteq X$
⟨*proof*⟩

lemma *eclose-least*:
$[\![Transset(X);\ A{<=}X]\!] \Longrightarrow eclose(A) \subseteq X$
⟨*proof*⟩

lemma *eclose-induct-down* [*consumes 1*]:
$[\![a \in eclose(b);$
$\bigwedge y.\ [\![y \in b]\!] \Longrightarrow P(y);$
$\bigwedge y\ z.\ [\![y \in eclose(b);\ P(y);\ z \in y]\!] \Longrightarrow P(z)$
$]\!] \Longrightarrow P(a)$
⟨*proof*⟩

lemma *Transset-eclose-eq-arg*: $Transset(X) \Longrightarrow eclose(X) = X$
⟨*proof*⟩

A transitive set either is empty or contains the empty set.

lemma *Transset-0-lemma* [*rule-format*]: $Transset(A) \Longrightarrow x{\in}A \longrightarrow 0{\in}A$
⟨*proof*⟩

lemma *Transset-0-disj*: $Transset(A) \Longrightarrow A{=}0 \mid 0{\in}A$
⟨*proof*⟩

18.3 Epsilon Recursion

lemma *mem-eclose-trans*: $[\![A \in eclose(B);\ \ B \in eclose(C)]\!] \Longrightarrow A \in eclose(C)$
⟨*proof*⟩

lemma *mem-eclose-sing-trans*:
$[\![A \in eclose(\{B\});\ \ B \in eclose(\{C\})]\!] \Longrightarrow A \in eclose(\{C\})$
⟨*proof*⟩

lemma *under-Memrel*: $[\![Transset(i);\ \ j \in i]\!] \Longrightarrow Memrel(i) -`` \{j\} = j$
⟨*proof*⟩

lemma *lt-Memrel*: $j < i \Longrightarrow Memrel(i) -`` \{j\} = j$
⟨*proof*⟩

lemmas *under-Memrel-eclose* = *Transset-eclose* [*THEN under-Memrel*]

lemmas *wfrec-ssubst* = *wf-Memrel* [*THEN wfrec*, *THEN ssubst*]

lemma *wfrec-eclose-eq*:
$[\![k \in eclose(\{j\});\ \ j \in eclose(\{i\})]\!] \Longrightarrow$
$wfrec(Memrel(eclose(\{i\})),\ k,\ H) = wfrec(Memrel(eclose(\{j\})),\ k,\ H)$
⟨*proof*⟩

lemma *wfrec-eclose-eq2*:
$k \in i \Longrightarrow wfrec(Memrel(eclose(\{i\})),k,H) = wfrec(Memrel(eclose(\{k\})),k,H)$
⟨*proof*⟩

lemma *transrec*: $transrec(a,H) = H(a,\ \lambda x{\in}a.\ transrec(x,H))$
⟨*proof*⟩

lemma *def-transrec*:
$[\![\bigwedge x.\ f(x){\equiv}transrec(x,H)]\!] \Longrightarrow f(a) = H(a,\ \lambda x{\in}a.\ f(x))$
⟨*proof*⟩

lemma *transrec-type*:
$[\![\bigwedge x\ u.\ [\![x \in eclose(\{a\});\ \ u \in Pi(x,B)]\!] \Longrightarrow H(x,u) \in B(x)]\!]$
$\Longrightarrow transrec(a,H) \in B(a)$
⟨*proof*⟩

lemma *eclose-sing-Ord*: $Ord(i) \Longrightarrow eclose(\{i\}) \subseteq succ(i)$
⟨*proof*⟩

lemma *succ-subset-eclose-sing*: $succ(i) \subseteq eclose(\{i\})$
⟨*proof*⟩

lemma *eclose-sing-Ord-eq*: $Ord(i) \Longrightarrow eclose(\{i\}) = succ(i)$

⟨*proof*⟩

lemma *Ord-transrec-type*:
assumes *jini*: $j \in i$
and *ordi*: $Ord(i)$
and *minor*: $\bigwedge x\ u.\ [\![x \in i;\ \ u \in Pi(x,B)]\!] \Longrightarrow H(x,u) \in B(x)$
shows $transrec(j,H) \in B(j)$
⟨*proof*⟩

18.4 Rank

lemma *rank*: $rank(a) = (\bigcup y{\in}a.\ succ(rank(y)))$
⟨*proof*⟩

lemma *Ord-rank* [*simp*]: $Ord(rank(a))$
⟨*proof*⟩

lemma *rank-of-Ord*: $Ord(i) \Longrightarrow rank(i) = i$
⟨*proof*⟩

lemma *rank-lt*: $a \in b \Longrightarrow rank(a) < rank(b)$
⟨*proof*⟩

lemma *eclose-rank-lt*: $a \in eclose(b) \Longrightarrow rank(a) < rank(b)$
⟨*proof*⟩

lemma *rank-mono*: $a{<=}b \Longrightarrow rank(a) \le rank(b)$
⟨*proof*⟩

lemma *rank-Pow*: $rank(Pow(a)) = succ(rank(a))$
⟨*proof*⟩

lemma *rank-0* [*simp*]: $rank(0) = 0$
⟨*proof*⟩

lemma *rank-succ* [*simp*]: $rank(succ(x)) = succ(rank(x))$
⟨*proof*⟩

lemma *rank-Union*: $rank(\bigcup(A)) = (\bigcup x{\in}A.\ rank(x))$
⟨*proof*⟩

lemma *rank-eclose*: $rank(eclose(a)) = rank(a)$
⟨*proof*⟩

lemma *rank-pair1*: $rank(a) < rank(\langle a,b\rangle)$
⟨*proof*⟩

lemma *rank-pair2*: $rank(b) < rank(\langle a,b\rangle)$
⟨*proof*⟩

lemma *the-equality-if*:
$P(a) \Longrightarrow (THE\ x.\ P(x)) = (if\ (\exists!x.\ P(x))\ then\ a\ else\ 0)$
⟨*proof*⟩

lemma *rank-apply*: $[\![i \in domain(f); function(f)]\!] \Longrightarrow rank(f`i) < rank(f)$
⟨*proof*⟩

18.5 Corollaries of Leastness

lemma *mem-eclose-subset*: $A \in B \Longrightarrow eclose(A) <= eclose(B)$
⟨*proof*⟩

lemma *eclose-mono*: $A <= B \Longrightarrow eclose(A) \subseteq eclose(B)$
⟨*proof*⟩

lemma *eclose-idem*: $eclose(eclose(A)) = eclose(A)$
⟨*proof*⟩

lemma *transrec2-0* [*simp*]: $transrec2(0,a,b) = a$
⟨*proof*⟩

lemma *transrec2-succ* [*simp*]: $transrec2(succ(i),a,b) = b(i,\ transrec2(i,a,b))$
⟨*proof*⟩

lemma *transrec2-Limit*:
$Limit(i) \Longrightarrow transrec2(i,a,b) = (\bigcup j<i.\ transrec2(j,a,b))$
⟨*proof*⟩

lemma *def-transrec2*:
$(\bigwedge x.\ f(x) \equiv transrec2(x,a,b))$
$\Longrightarrow f(0) = a \wedge$
$f(succ(i)) = b(i,\ f(i)) \wedge$
$(Limit(K) \longrightarrow f(K) = (\bigcup j<K.\ f(j)))$
⟨*proof*⟩

lemmas *recursor-lemma* = *recursor-def* [*THEN def-transrec*, *THEN trans*]

lemma *recursor-0*: $recursor(a,b,0) = a$

⟨*proof*⟩

lemma *recursor-succ*: $recursor(a,b,succ(m)) = b(m,\ recursor(a,b,m))$
⟨*proof*⟩

lemma *rec-0* [*simp*]: $rec(0,a,b) = a$
⟨*proof*⟩

lemma *rec-succ* [*simp*]: $rec(succ(m),a,b) = b(m,\ rec(m,a,b))$
⟨*proof*⟩

lemma *rec-type*:
$[\![n \in nat;$
$a \in C(0);$
$\bigwedge m\ z.\ [\![m \in nat;\ \ z \in C(m)]\!] \Longrightarrow b(m,z):\ C(succ(m))]\!]$
$\Longrightarrow rec(n,a,b) \in C(n)$
⟨*proof*⟩

end

19 Partial and Total Orderings: Basic Definitions and Properties

theory *Order* **imports** *WF Perm* **begin**

We adopt the following convention: *ord* is used for strict orders and *order* is used for their reflexive counterparts.

definition
$part\text{-}ord :: [i,i]\Rightarrow o$ **where**
$part\text{-}ord(A,r) \equiv irrefl(A,r) \wedge trans[A](r)$

definition
$linear :: [i,i]\Rightarrow o$ **where**
$linear(A,r) \equiv (\forall x{\in}A.\ \forall y{\in}A.\ \langle x,y\rangle{:}r \mid x{=}y \mid \langle y,x\rangle{:}r)$

definition
$tot\text{-}ord :: [i,i]\Rightarrow o$ **where**
$tot\text{-}ord(A,r) \equiv part\text{-}ord(A,r) \wedge linear(A,r)$

definition
$preorder\text{-}on(A,\ r) \equiv refl(A,\ r) \wedge trans[A](r)$

definition
$partial\text{-}order\text{-}on(A,\ r) \equiv preorder\text{-}on(A,\ r) \wedge antisym(r)$

abbreviation
$Preorder(r) \equiv preorder\text{-}on(field(r),\ r)$

abbreviation
$Partial\text{-}order(r) \equiv partial\text{-}order\text{-}on(field(r),\ r)$

definition
$well\text{-}ord :: [i,i]\Rightarrow o$ **where**
$well\text{-}ord(A,r) \equiv tot\text{-}ord(A,r) \wedge wf[A](r)$

definition
$mono\text{-}map :: [i,i,i,i]\Rightarrow i$ **where**
$mono\text{-}map(A,r,B,s) \equiv$
$\{f \in A->B.\ \forall x{\in}A.\ \forall y{\in}A.\ \langle x,y\rangle{:}r \longrightarrow <f`x,f`y>{:}s\}$

definition
$ord\text{-}iso :: [i,i,i,i]\Rightarrow i$ $(\langle(\langle\text{-},\ \text{-}\rangle \cong/\ \langle\text{-},\ \text{-}\rangle)\rangle\ 51)$ **where**
$\langle A,r\rangle \cong \langle B,s\rangle \equiv$
$\{f \in bij(A,B).\ \forall x{\in}A.\ \forall y{\in}A.\ \langle x,y\rangle{:}r \longleftrightarrow <f`x,f`y>{:}s\}$

definition
$pred :: [i,i,i]\Rightarrow i$ **where**
$pred(A,x,r) \equiv \{y \in A.\ \langle y,x\rangle{:}r\}$

definition
$ord\text{-}iso\text{-}map :: [i,i,i,i]\Rightarrow i$ **where**
$ord\text{-}iso\text{-}map(A,r,B,s) \equiv$
$\bigcup x{\in}A.\ \bigcup y{\in}B.\ \bigcup f \in ord\text{-}iso(pred(A,x,r),\ r,\ pred(B,y,s),\ s).\ \{\langle x,y\rangle\}$

definition
$first :: [i,\ i,\ i] \Rightarrow o$ **where**
$first(u,\ X,\ R) \equiv u \in X \wedge (\forall v{\in}X.\ v{\neq}u \longrightarrow \langle u,v\rangle \in R)$

19.1 Immediate Consequences of the Definitions

lemma *part-ord-Imp-asym*:
$part\text{-}ord(A,r) \Longrightarrow asym(r \cap A{*}A)$
⟨*proof*⟩

lemma *linearE*:
$[\![linear(A,r);\ \ x \in A;\ \ y \in A;$
$\langle x,y\rangle{:}r \Longrightarrow P;\ \ x{=}y \Longrightarrow P;\ \ \langle y,x\rangle{:}r \Longrightarrow P]\!]$
$\Longrightarrow P$
⟨*proof*⟩

lemma *well-ordI*:

$[\![wf[A](r);\ linear(A,r)]\!] \Longrightarrow well\text{-}ord(A,r)$
⟨*proof*⟩

lemma *well-ord-is-wf*:
$well\text{-}ord(A,r) \Longrightarrow wf[A](r)$
⟨*proof*⟩

lemma *well-ord-is-trans-on*:
$well\text{-}ord(A,r) \Longrightarrow trans[A](r)$
⟨*proof*⟩

lemma *well-ord-is-linear*: $well\text{-}ord(A,r) \Longrightarrow linear(A,r)$
⟨*proof*⟩

lemma *pred-iff*: $y \in pred(A,x,r) \longleftrightarrow \langle y,x\rangle{:}r \wedge y \in A$
⟨*proof*⟩

lemmas *predI* = *conjI* [*THEN pred-iff* [*THEN iffD2*]]

lemma *predE*: $[\![y \in pred(A,x,r);\ [\![y \in A;\ \langle y,x\rangle{:}r]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *pred-subset-under*: $pred(A,x,r) \subseteq r\ -``\ \{x\}$
⟨*proof*⟩

lemma *pred-subset*: $pred(A,x,r) \subseteq A$
⟨*proof*⟩

lemma *pred-pred-eq*:
$pred(pred(A,x,r),\ y,\ r) = pred(A,x,r) \cap pred(A,y,r)$
⟨*proof*⟩

lemma *trans-pred-pred-eq*:
$[\![trans[A](r);\ \langle y,x\rangle{:}r;\ x \in A;\ y \in A]\!]$
$\Longrightarrow pred(pred(A,x,r),\ y,\ r) = pred(A,y,r)$
⟨*proof*⟩

19.2 Restricting an Ordering's Domain

lemma *part-ord-subset*:
$[\![part\text{-}ord(A,r);\ B<=A]\!] \Longrightarrow part\text{-}ord(B,r)$
⟨*proof*⟩

lemma *linear-subset*:
$[\![linear(A,r);\ B<=A]\!] \Longrightarrow linear(B,r)$
⟨*proof*⟩

lemma *tot-ord-subset*:
⟦*tot-ord*(*A*,*r*); *B*<=*A*⟧ ⟹ *tot-ord*(*B*,*r*)
⟨*proof*⟩

lemma *well-ord-subset*:
⟦*well-ord*(*A*,*r*); *B*<=*A*⟧ ⟹ *well-ord*(*B*,*r*)
⟨*proof*⟩

lemma *irrefl-Int-iff*: *irrefl*(*A*,*r* ∩ *A*∗*A*) ⟷ *irrefl*(*A*,*r*)
⟨*proof*⟩

lemma *trans-on-Int-iff*: *trans*[*A*](*r* ∩ *A*∗*A*) ⟷ *trans*[*A*](*r*)
⟨*proof*⟩

lemma *part-ord-Int-iff*: *part-ord*(*A*,*r* ∩ *A*∗*A*) ⟷ *part-ord*(*A*,*r*)
⟨*proof*⟩

lemma *linear-Int-iff*: *linear*(*A*,*r* ∩ *A*∗*A*) ⟷ *linear*(*A*,*r*)
⟨*proof*⟩

lemma *tot-ord-Int-iff*: *tot-ord*(*A*,*r* ∩ *A*∗*A*) ⟷ *tot-ord*(*A*,*r*)
⟨*proof*⟩

lemma *wf-on-Int-iff*: *wf*[*A*](*r* ∩ *A*∗*A*) ⟷ *wf*[*A*](*r*)
⟨*proof*⟩

lemma *well-ord-Int-iff*: *well-ord*(*A*,*r* ∩ *A*∗*A*) ⟷ *well-ord*(*A*,*r*)
⟨*proof*⟩

19.3 Empty and Unit Domains

lemma *wf-on-any-0*: *wf*[*A*](*0*)
⟨*proof*⟩

19.3.1 Relations over the Empty Set

lemma *irrefl-0*: *irrefl*(*0*,*r*)
⟨*proof*⟩

lemma *trans-on-0*: *trans*[*0*](*r*)
⟨*proof*⟩

lemma *part-ord-0*: *part-ord*(*0*,*r*)
⟨*proof*⟩

lemma *linear-0*: *linear*(*0*,*r*)

⟨*proof*⟩

lemma *tot-ord-0*: *tot-ord*(*0*,*r*)
⟨*proof*⟩

lemma *wf-on-0*: *wf*[*0*](*r*)
⟨*proof*⟩

lemma *well-ord-0*: *well-ord*(*0*,*r*)
⟨*proof*⟩

19.3.2 The Empty Relation Well-Orders the Unit Set

by Grabczewski

lemma *tot-ord-unit*: *tot-ord*({*a*},*0*)
⟨*proof*⟩

lemma *well-ord-unit*: *well-ord*({*a*},*0*)
⟨*proof*⟩

19.4 Order-Isomorphisms

Suppes calls them "similarities"

lemma *mono-map-is-fun*: $f \in mono\text{-}map(A,r,B,s) \Longrightarrow f \in A->B$
⟨*proof*⟩

lemma *mono-map-is-inj*:
$[\![linear(A,r);\ \ wf[B](s);\ \ f \in mono\text{-}map(A,r,B,s)]\!] \Longrightarrow f \in inj(A,B)$
⟨*proof*⟩

lemma *ord-isoI*:
$[\![f \in bij(A,\ B);$
$\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow \langle x,\ y\rangle \in r \longleftrightarrow <f`x,\ f`y> \in s]\!]$
$\Longrightarrow f \in ord\text{-}iso(A,r,B,s)$
⟨*proof*⟩

lemma *ord-iso-is-mono-map*:
$f \in ord\text{-}iso(A,r,B,s) \Longrightarrow f \in mono\text{-}map(A,r,B,s)$
⟨*proof*⟩

lemma *ord-iso-is-bij*:
$f \in ord\text{-}iso(A,r,B,s) \Longrightarrow f \in bij(A,B)$
⟨*proof*⟩

lemma *ord-iso-apply*:
$[\![f \in ord\text{-}iso(A,r,B,s);\ \ \langle x,y\rangle\colon r;\ \ x \in A;\ \ y \in A]\!] \Longrightarrow <f`x,\ f`y> \in s$
⟨*proof*⟩

lemma *ord-iso-converse*:
$[\![f \in \mathit{ord\text{-}iso}(A,r,B,s);\ \langle x,y\rangle\colon s;\ x \in B;\ y \in B]\!]$
$\Longrightarrow <\mathit{converse}(f)\ `\ x,\ \mathit{converse}(f)\ `\ y> \in r$
⟨*proof*⟩

lemma *ord-iso-refl*: $\mathit{id}(A)\colon \mathit{ord\text{-}iso}(A,r,A,r)$
⟨*proof*⟩

lemma *ord-iso-sym*: $f \in \mathit{ord\text{-}iso}(A,r,B,s) \Longrightarrow \mathit{converse}(f)\colon \mathit{ord\text{-}iso}(B,s,A,r)$
⟨*proof*⟩

lemma *mono-map-trans*:
$[\![g \in \mathit{mono\text{-}map}(A,r,B,s);\ f \in \mathit{mono\text{-}map}(B,s,C,t)]\!]$
$\Longrightarrow (f\ O\ g)\colon \mathit{mono\text{-}map}(A,r,C,t)$
⟨*proof*⟩

lemma *ord-iso-trans*:
$[\![g \in \mathit{ord\text{-}iso}(A,r,B,s);\ f \in \mathit{ord\text{-}iso}(B,s,C,t)]\!]$
$\Longrightarrow (f\ O\ g)\colon \mathit{ord\text{-}iso}(A,r,C,t)$
⟨*proof*⟩

lemma *mono-ord-isoI*:
$[\![f \in \mathit{mono\text{-}map}(A,r,B,s);\ g \in \mathit{mono\text{-}map}(B,s,A,r);$
$f\ O\ g = \mathit{id}(B);\ g\ O\ f = \mathit{id}(A)]\!] \Longrightarrow f \in \mathit{ord\text{-}iso}(A,r,B,s)$
⟨*proof*⟩

lemma *well-ord-mono-ord-isoI*:
$[\![\mathit{well\text{-}ord}(A,r);\ \mathit{well\text{-}ord}(B,s);$
$f \in \mathit{mono\text{-}map}(A,r,B,s);\ \mathit{converse}(f)\colon \mathit{mono\text{-}map}(B,s,A,r)]\!]$
$\Longrightarrow f \in \mathit{ord\text{-}iso}(A,r,B,s)$
⟨*proof*⟩

lemma *part-ord-ord-iso*:
$[\![\mathit{part\text{-}ord}(B,s);\ f \in \mathit{ord\text{-}iso}(A,r,B,s)]\!] \Longrightarrow \mathit{part\text{-}ord}(A,r)$
⟨*proof*⟩

lemma *linear-ord-iso*:
$[\![linear(B,s);\ f \in ord\text{-}iso(A,r,B,s)]\!] \Longrightarrow linear(A,r)$
⟨*proof*⟩

lemma *wf-on-ord-iso*:
$[\![wf[B](s);\ f \in ord\text{-}iso(A,r,B,s)]\!] \Longrightarrow wf[A](r)$
⟨*proof*⟩

lemma *well-ord-ord-iso*:
$[\![well\text{-}ord(B,s);\ f \in ord\text{-}iso(A,r,B,s)]\!] \Longrightarrow well\text{-}ord(A,r)$
⟨*proof*⟩

19.5 Main results of Kunen, Chapter 1 section 6

lemma *well-ord-iso-subset-lemma*:
$[\![well\text{-}ord(A,r);\ f \in ord\text{-}iso(A,r,\ A',r);\ A'<=\ A;\ y \in A]\!]$
$\Longrightarrow \neg\ <f`y,\ y>:\ r$
⟨*proof*⟩

lemma *well-ord-iso-predE*:
$[\![well\text{-}ord(A,r);\ f \in ord\text{-}iso(A,\ r,\ pred(A,x,r),\ r);\ x \in A]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *well-ord-iso-pred-eq*:
$[\![well\text{-}ord(A,r);\ f \in ord\text{-}iso(pred(A,a,r),\ r,\ pred(A,c,r),\ r);$
$a \in A;\ c \in A]\!] \Longrightarrow a=c$
⟨*proof*⟩

lemma *ord-iso-image-pred*:
$[\![f \in ord\text{-}iso(A,r,B,s);\ a \in A]\!] \Longrightarrow f\ ``\ pred(A,a,r) = pred(B,\ f`a,\ s)$
⟨*proof*⟩

lemma *ord-iso-restrict-image*:
$[\![f \in ord\text{-}iso(A,r,B,s);\ C<=A]\!]$
$\Longrightarrow restrict(f,C) \in ord\text{-}iso(C,\ r,\ f``C,\ s)$
⟨*proof*⟩

lemma *ord-iso-restrict-pred*:
$[\![f \in ord\text{-}iso(A,r,B,s);\ \ a \in A]\!]$
$\Longrightarrow restrict(f,\ pred(A,a,r)) \in ord\text{-}iso(pred(A,a,r),\ r,\ pred(B,\ f`a,\ s),\ s)$
⟨*proof*⟩

lemma *well-ord-iso-preserving*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s);\ \langle a,c\rangle:\ r;$

$f \in$ *ord-iso*(*pred*(*A*,*a*,*r*), *r*, *pred*(*B*,*b*,*s*), *s*);
$g \in$ *ord-iso*(*pred*(*A*,*c*,*r*), *r*, *pred*(*B*,*d*,*s*), *s*);
$a \in A$; $c \in A$; $b \in B$; $d \in B$⟧ ⟹ ⟨*b*,*d*⟩: *s*
⟨*proof*⟩

lemma *well-ord-iso-unique-lemma*:
⟦*well-ord*(*A*,*r*);
$f \in$ *ord-iso*(*A*,*r*, *B*,*s*); $g \in$ *ord-iso*(*A*,*r*, *B*,*s*); $y \in A$⟧
⟹ ¬ <*g*‘*y*, *f*‘*y*> ∈ *s*
⟨*proof*⟩

lemma *well-ord-iso-unique*: ⟦*well-ord*(*A*,*r*);
$f \in$ *ord-iso*(*A*,*r*, *B*,*s*); $g \in$ *ord-iso*(*A*,*r*, *B*,*s*)⟧ ⟹ $f = g$
⟨*proof*⟩

19.6 Towards Kunen's Theorem 6.3: Linearity of the Similarity Relation

lemma *ord-iso-map-subset*: *ord-iso-map*(*A*,*r*,*B*,*s*) ⊆ *A*∗*B*
⟨*proof*⟩

lemma *domain-ord-iso-map*: *domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)) ⊆ *A*
⟨*proof*⟩

lemma *range-ord-iso-map*: *range*(*ord-iso-map*(*A*,*r*,*B*,*s*)) ⊆ *B*
⟨*proof*⟩

lemma *converse-ord-iso-map*:
converse(*ord-iso-map*(*A*,*r*,*B*,*s*)) = *ord-iso-map*(*B*,*s*,*A*,*r*)
⟨*proof*⟩

lemma *function-ord-iso-map*:
well-ord(*B*,*s*) ⟹ *function*(*ord-iso-map*(*A*,*r*,*B*,*s*))
⟨*proof*⟩

lemma *ord-iso-map-fun*: *well-ord*(*B*,*s*) ⟹ *ord-iso-map*(*A*,*r*,*B*,*s*)
∈ *domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)) −> *range*(*ord-iso-map*(*A*,*r*,*B*,*s*))
⟨*proof*⟩

lemma *ord-iso-map-mono-map*:
⟦*well-ord*(*A*,*r*); *well-ord*(*B*,*s*)⟧
⟹ *ord-iso-map*(*A*,*r*,*B*,*s*)
∈ *mono-map*(*domain*(*ord-iso-map*(*A*,*r*,*B*,*s*)), *r*,
range(*ord-iso-map*(*A*,*r*,*B*,*s*)), *s*)
⟨*proof*⟩

lemma *ord-iso-map-ord-iso*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s)]\!] \Longrightarrow ord\text{-}iso\text{-}map(A,r,B,s) \in ord\text{-}iso(domain(ord\text{-}iso\text{-}map(A,r,B,s)),\ r,\ range(ord\text{-}iso\text{-}map(A,r,B,s)),\ s)$
⟨*proof*⟩

lemma *domain-ord-iso-map-subset*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s);\ a \in A;\ a \notin domain(ord\text{-}iso\text{-}map(A,r,B,s))]\!] \Longrightarrow domain(ord\text{-}iso\text{-}map(A,r,B,s)) \subseteq pred(A,\ a,\ r)$
⟨*proof*⟩

lemma *domain-ord-iso-map-cases*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s)]\!] \Longrightarrow domain(ord\text{-}iso\text{-}map(A,r,B,s)) = A \mid (\exists\, x{\in}A.\ domain(ord\text{-}iso\text{-}map(A,r,B,s)) = pred(A,x,r))$
⟨*proof*⟩

lemma *range-ord-iso-map-cases*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s)]\!] \Longrightarrow range(ord\text{-}iso\text{-}map(A,r,B,s)) = B \mid (\exists\, y{\in}B.\ range(ord\text{-}iso\text{-}map(A,r,B,s)) = pred(B,y,s))$
⟨*proof*⟩

Kunen's Theorem 6.3: Fundamental Theorem for Well-Ordered Sets

theorem *well-ord-trichotomy*:
$[\![well\text{-}ord(A,r);\ well\text{-}ord(B,s)]\!] \Longrightarrow ord\text{-}iso\text{-}map(A,r,B,s) \in ord\text{-}iso(A,\ r,\ B,\ s) \mid (\exists\, x{\in}A.\ ord\text{-}iso\text{-}map(A,r,B,s) \in ord\text{-}iso(pred(A,x,r),\ r,\ B,\ s)) \mid (\exists\, y{\in}B.\ ord\text{-}iso\text{-}map(A,r,B,s) \in ord\text{-}iso(A,\ r,\ pred(B,y,s),\ s))$
⟨*proof*⟩

19.7 Miscellaneous Results by Krzysztof Grabczewski

lemma *irrefl-converse*: $irrefl(A,r) \Longrightarrow irrefl(A,converse(r))$
⟨*proof*⟩

lemma *trans-on-converse*: $trans[A](r) \Longrightarrow trans[A](converse(r))$
⟨*proof*⟩

lemma *part-ord-converse*: $part\text{-}ord(A,r) \Longrightarrow part\text{-}ord(A,converse(r))$
⟨*proof*⟩

lemma *linear-converse*: $linear(A,r) \Longrightarrow linear(A,converse(r))$
⟨*proof*⟩

lemma *tot-ord-converse*: *tot-ord*(A,r) $\Longrightarrow$ *tot-ord*(A,*converse*(r))
⟨*proof*⟩

lemma *first-is-elem*: *first*(b,B,r) $\Longrightarrow$ $b \in B$
⟨*proof*⟩

lemma *well-ord-imp-ex1-first*:
$[\![$*well-ord*(A,r); B<=A; $B \neq 0]\!] \Longrightarrow (\exists!b.$ *first*(b,B,r))
⟨*proof*⟩

lemma *the-first-in*:
$[\![$*well-ord*(A,r); B<=A; $B \neq 0]\!] \Longrightarrow$ (*THE* $b.$ *first*(b,B,r)) $\in B$
⟨*proof*⟩

19.8 Lemmas for the Reflexive Orders

lemma *subset-vimage-vimage-iff*:
$[\![$*Preorder*(r); $A \subseteq$ *field*(r); $B \subseteq$ *field*(r)$]\!] \Longrightarrow$
$r -`` A \subseteq r -`` B \longleftrightarrow (\forall a{\in}A.\ \exists b{\in}B.\ \langle a,\ b\rangle \in r)$
⟨*proof*⟩

lemma *subset-vimage1-vimage1-iff*:
$[\![$*Preorder*(r); $a \in$ *field*(r); $b \in$ *field*(r)$]\!] \Longrightarrow$
$r -`` \{a\} \subseteq r -`` \{b\} \longleftrightarrow \langle a,\ b\rangle \in r$
⟨*proof*⟩

lemma *Refl-antisym-eq-Image1-Image1-iff*:
$[\![$*refl*(*field*(r), r); *antisym*(r); $a \in$ *field*(r); $b \in$ *field*(r)$]\!] \Longrightarrow$
$r `` \{a\} = r `` \{b\} \longleftrightarrow a = b$
⟨*proof*⟩

lemma *Partial-order-eq-Image1-Image1-iff*:
$[\![$*Partial-order*(r); $a \in$ *field*(r); $b \in$ *field*(r)$]\!] \Longrightarrow$
$r `` \{a\} = r `` \{b\} \longleftrightarrow a = b$
⟨*proof*⟩

lemma *Refl-antisym-eq-vimage1-vimage1-iff*:
$[\![$*refl*(*field*(r), r); *antisym*(r); $a \in$ *field*(r); $b \in$ *field*(r)$]\!] \Longrightarrow$
$r -`` \{a\} = r -`` \{b\} \longleftrightarrow a = b$
⟨*proof*⟩

lemma *Partial-order-eq-vimage1-vimage1-iff*:
$[\![$*Partial-order*(r); $a \in$ *field*(r); $b \in$ *field*(r)$]\!] \Longrightarrow$
$r -`` \{a\} = r -`` \{b\} \longleftrightarrow a = b$
⟨*proof*⟩

end

20 Combining Orderings: Foundations of Ordinal Arithmetic

theory *OrderArith* **imports** *Order Sum Ordinal* **begin**

definition

$$\begin{array}{l} radd \quad :: [i,i,i,i]\Rightarrow i \ \textbf{where} \\ \quad radd(A,r,B,s) \equiv \\ \qquad \{z{:}\ (A+B) * (A+B). \\ \qquad\quad (\exists\, x\ y.\ z = \langle Inl(x),\ Inr(y)\rangle) \quad | \\ \qquad\quad (\exists\, x'\ x.\ z = \langle Inl(x'),\ Inl(x)\rangle \wedge \langle x',x\rangle{:}r) \quad | \\ \qquad\quad (\exists\, y'\ y.\ z = \langle Inr(y'),\ Inr(y)\rangle \wedge \langle y',y\rangle{:}s)\} \end{array}$$

definition

$$\begin{array}{l} rmult \quad :: [i,i,i,i]\Rightarrow i \ \textbf{where} \\ \quad rmult(A,r,B,s) \equiv \\ \qquad \{z{:}\ (A*B) * (A*B). \\ \qquad\quad \exists\, x'\ y'\ x\ y.\ z = \langle\langle x',y'\rangle, \langle x,y\rangle\rangle \wedge \\ \qquad\qquad (\langle x',x\rangle{:}\ r \mid (x'{=}x \wedge \langle y',y\rangle{:}\ s))\} \end{array}$$

definition

$$\begin{array}{l} rvimage :: [i,i,i]\Rightarrow i \ \textbf{where} \\ \quad rvimage(A,f,r) \equiv \{z \in A*A.\ \exists\, x\ y.\ z = \langle x,y\rangle \wedge \langle f`x,f`y\rangle{:}\ r\} \end{array}$$

definition

$$\begin{array}{l} measure :: [i,\ i\Rightarrow i] \Rightarrow i \ \textbf{where} \\ \quad measure(A,f) \equiv \{\langle x,y\rangle{:}\ A*A.\ f(x) < f(y)\} \end{array}$$

20.1 Addition of Relations – Disjoint Sum

20.1.1 Rewrite rules. Can be used to obtain introduction rules

lemma *radd-Inl-Inr-iff* [*iff*]:
$\langle Inl(a),\ Inr(b)\rangle \in radd(A,r,B,s) \ \longleftrightarrow\ a \in A \wedge b \in B$
⟨*proof*⟩

lemma *radd-Inl-iff* [*iff*]:
$\langle Inl(a'),\ Inl(a)\rangle \in radd(A,r,B,s) \ \longleftrightarrow\ a'{:}A \wedge a \in A \wedge \langle a',a\rangle{:}r$
⟨*proof*⟩

lemma *radd-Inr-iff* [*iff*]:
$\langle Inr(b'),\ Inr(b)\rangle \in radd(A,r,B,s) \longleftrightarrow\ b'{:}B \wedge b \in B \wedge \langle b',b\rangle{:}s$
⟨*proof*⟩

lemma *radd-Inr-Inl-iff* [*simp*]:
$\langle Inr(b),\ Inl(a)\rangle \in radd(A,r,B,s) \longleftrightarrow False$
⟨*proof*⟩

declare *radd-Inr-Inl-iff* [*THEN iffD1*, *dest*!]

20.1.2 Elimination Rule

lemma *raddE*:
$[\![\langle p',p\rangle \in radd(A,r,B,s);$
$\bigwedge x\ y.\ [\![p'=Inl(x);\ x \in A;\ p=Inr(y);\ y \in B]\!] \Longrightarrow Q;$
$\bigwedge x'\ x.\ [\![p'=Inl(x');\ p=Inl(x);\ \langle x',x\rangle : r;\ x':A;\ x \in A]\!] \Longrightarrow Q;$
$\bigwedge y'\ y.\ [\![p'=Inr(y');\ p=Inr(y);\ \langle y',y\rangle : s;\ y':B;\ y \in B]\!] \Longrightarrow Q$
$]\!] \Longrightarrow Q$
⟨*proof*⟩

20.1.3 Type checking

lemma *radd-type*: $radd(A,r,B,s) \subseteq (A+B) * (A+B)$
⟨*proof*⟩

lemmas *field-radd* = *radd-type* [*THEN field-rel-subset*]

20.1.4 Linearity

lemma *linear-radd*:
$[\![linear(A,r);\ \ linear(B,s)]\!] \Longrightarrow linear(A+B,radd(A,r,B,s))$
⟨*proof*⟩

20.1.5 Well-foundedness

lemma *wf-on-radd*: $[\![wf[A](r);\ \ wf[B](s)]\!] \Longrightarrow wf[A+B](radd(A,r,B,s))$
⟨*proof*⟩

lemma *wf-radd*: $[\![wf(r);\ \ wf(s)]\!] \Longrightarrow wf(radd(field(r),r,field(s),s))$
⟨*proof*⟩

lemma *well-ord-radd*:
$[\![well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!] \Longrightarrow well\text{-}ord(A+B,\ radd(A,r,B,s))$
⟨*proof*⟩

20.1.6 An *ord-iso* congruence law

lemma *sum-bij*:
$[\![f \in bij(A,C);\ \ g \in bij(B,D)]\!]$
$\Longrightarrow (\lambda z{\in}A+B.\ case(\lambda x.\ Inl(f`x),\ \lambda y.\ Inr(g`y),\ z)) \in bij(A+B,\ C+D)$
⟨*proof*⟩

lemma *sum-ord-iso-cong*:

$[\![f \in \mathit{ord\text{-}iso}(A,r,A',r');\ \ g \in \mathit{ord\text{-}iso}(B,s,B',s')]\!] \Longrightarrow$
$(\lambda z{\in}A{+}B.\ \mathit{case}(\lambda x.\ \mathit{Inl}(f`x),\ \lambda y.\ \mathit{Inr}(g`y),\ z))$
$\in \mathit{ord\text{-}iso}(A{+}B,\ \mathit{radd}(A,r,B,s),\ A'{+}B',\ \mathit{radd}(A',r',B',s'))$
⟨*proof*⟩

lemma *sum-disjoint-bij*: $A \cap B = 0 \Longrightarrow$
$(\lambda z{\in}A{+}B.\ \mathit{case}(\lambda x.\ x,\ \lambda y.\ y,\ z)) \in \mathit{bij}(A{+}B,\ A \cup B)$
⟨*proof*⟩

20.1.7 Associativity

lemma *sum-assoc-bij*:
$(\lambda z{\in}(A{+}B){+}C.\ \mathit{case}(\mathit{case}(\mathit{Inl},\ \lambda y.\ \mathit{Inr}(\mathit{Inl}(y))),\ \lambda y.\ \mathit{Inr}(\mathit{Inr}(y)),\ z))$
$\in \mathit{bij}((A{+}B){+}C,\ A{+}(B{+}C))$
⟨*proof*⟩

lemma *sum-assoc-ord-iso*:
$(\lambda z{\in}(A{+}B){+}C.\ \mathit{case}(\mathit{case}(\mathit{Inl},\ \lambda y.\ \mathit{Inr}(\mathit{Inl}(y))),\ \lambda y.\ \mathit{Inr}(\mathit{Inr}(y)),\ z))$
$\in \mathit{ord\text{-}iso}((A{+}B){+}C,\ \mathit{radd}(A{+}B,\ \mathit{radd}(A,r,B,s),\ C,\ t),$
$A{+}(B{+}C),\ \mathit{radd}(A,\ r,\ B{+}C,\ \mathit{radd}(B,s,C,t)))$
⟨*proof*⟩

20.2 Multiplication of Relations – Lexicographic Product

20.2.1 Rewrite rule. Can be used to obtain introduction rules

lemma *rmult-iff* [*iff*]:
$\langle\langle a',b'\rangle,\ \langle a,b\rangle\rangle \in \mathit{rmult}(A,r,B,s) \longleftrightarrow$
$(\langle a',a\rangle\colon r\ \wedge\ a'{:}A \wedge a \in A \wedge b'\colon B \wedge b \in B)\ |$
$(\langle b',b\rangle\colon s\ \wedge\ a'{=}a \wedge a \in A \wedge b'\colon B \wedge b \in B)$

⟨*proof*⟩

lemma *rmultE*:
$[\![\langle\langle a',b'\rangle,\ \langle a,b\rangle\rangle \in \mathit{rmult}(A,r,B,s);$
$[\![\langle a',a\rangle\colon r;\ \ a'{:}A;\ \ a \in A;\ \ b'{:}B;\ \ b \in B]\!] \Longrightarrow Q;$
$[\![\langle b',b\rangle\colon s;\ \ a \in A;\ \ a'{=}a;\ \ b'{:}B;\ \ b \in B]\!] \Longrightarrow Q$
$]\!] \Longrightarrow Q$
⟨*proof*⟩

20.2.2 Type checking

lemma *rmult-type*: $\mathit{rmult}(A,r,B,s) \subseteq (A{*}B) * (A{*}B)$
⟨*proof*⟩

lemmas *field-rmult* = *rmult-type* [*THEN field-rel-subset*]

20.2.3 Linearity

lemma *linear-rmult*:
$[\![linear(A,r);\ \ linear(B,s)]\!] \Longrightarrow linear(A*B,rmult(A,r,B,s))$
⟨*proof*⟩

20.2.4 Well-foundedness

lemma *wf-on-rmult*: $[\![wf[A](r);\ \ wf[B](s)]\!] \Longrightarrow wf[A*B](rmult(A,r,B,s))$
⟨*proof*⟩

lemma *wf-rmult*: $[\![wf(r);\ \ wf(s)]\!] \Longrightarrow wf(rmult(field(r),r,field(s),s))$
⟨*proof*⟩

lemma *well-ord-rmult*:
$[\![well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!] \Longrightarrow well\text{-}ord(A*B,\ rmult(A,r,B,s))$
⟨*proof*⟩

20.2.5 An *ord-iso* congruence law

lemma *prod-bij*:
$[\![f \in bij(A,C);\ \ g \in bij(B,D)]\!]$
$\Longrightarrow (lam\ \langle x,y\rangle{:}A*B.\ \langle f`x,\ g`y\rangle) \in bij(A*B,\ C*D)$
⟨*proof*⟩

lemma *prod-ord-iso-cong*:
$[\![f \in ord\text{-}iso(A,r,A',r');\ \ g \in ord\text{-}iso(B,s,B',s')]\!]$
$\Longrightarrow (lam\ \langle x,y\rangle{:}A*B.\ \langle f`x,\ g`y\rangle)$
$\in ord\text{-}iso(A*B,\ rmult(A,r,B,s),\ A'*B',\ rmult(A',r',B',s'))$
⟨*proof*⟩

lemma *singleton-prod-bij*: $(\lambda z{\in}A.\ \langle x,z\rangle) \in bij(A,\ \{x\}*A)$
⟨*proof*⟩

lemma *singleton-prod-ord-iso*:
$well\text{-}ord(\{x\},xr) \Longrightarrow$
$(\lambda z{\in}A.\ \langle x,z\rangle) \in ord\text{-}iso(A,\ r,\ \{x\}*A,\ rmult(\{x\},\ xr,\ A,\ r))$
⟨*proof*⟩

lemma *prod-sum-singleton-bij*:
$a \notin C \Longrightarrow$
$(\lambda x{\in}C*B + D.\ case(\lambda x.\ x,\ \lambda y.\langle a,y\rangle,\ x))$
$\in bij(C*B + D,\ C*B \cup \{a\}*D)$
⟨*proof*⟩

lemma *prod-sum-singleton-ord-iso*:
$[\![a \in A;\ \ well\text{-}ord(A,r)]\!] \Longrightarrow$

$(\lambda x{\in}pred(A,a,r){*}B + pred(B,b,s).\ case(\lambda x.\ x,\ \lambda y.\langle a,y\rangle,\ x))$
$\in ord\text{-}iso(pred(A,a,r){*}B + pred(B,b,s),$
$radd(A{*}B,\ rmult(A,r,B,s),\ B,\ s),$
$pred(A,a,r){*}B \cup \{a\}{*}pred(B,b,s),\ rmult(A,r,B,s))$
⟨*proof*⟩

20.2.6 Distributive law

lemma *sum-prod-distrib-bij*:
$(lam\ \langle x,z\rangle{:}(A+B){*}C.\ case(\lambda y.\ Inl(\langle y,z\rangle),\ \lambda y.\ Inr(\langle y,z\rangle),\ x))$
$\in bij((A+B){*}C,\ (A{*}C)+(B{*}C))$
⟨*proof*⟩

lemma *sum-prod-distrib-ord-iso*:
$(lam\ \langle x,z\rangle{:}(A+B){*}C.\ case(\lambda y.\ Inl(\langle y,z\rangle),\ \lambda y.\ Inr(\langle y,z\rangle),\ x))$
$\in ord\text{-}iso((A+B){*}C,\ rmult(A+B,\ radd(A,r,B,s),\ C,\ t),$
$(A{*}C)+(B{*}C),\ radd(A{*}C,\ rmult(A,r,C,t),\ B{*}C,\ rmult(B,s,C,t)))$
⟨*proof*⟩

20.2.7 Associativity

lemma *prod-assoc-bij*:
$(lam\ \langle\langle x,y\rangle,\ z\rangle{:}(A{*}B){*}C.\ \langle x,\langle y,z\rangle\rangle) \in bij((A{*}B){*}C,\ A{*}(B{*}C))$
⟨*proof*⟩

lemma *prod-assoc-ord-iso*:
$(lam\ \langle\langle x,y\rangle,\ z\rangle{:}(A{*}B){*}C.\ \langle x,\langle y,z\rangle\rangle)$
$\in ord\text{-}iso((A{*}B){*}C,\ rmult(A{*}B,\ rmult(A,r,B,s),\ C,\ t),$
$A{*}(B{*}C),\ rmult(A,\ r,\ B{*}C,\ rmult(B,s,C,t)))$
⟨*proof*⟩

20.3 Inverse Image of a Relation

20.3.1 Rewrite rule

lemma *rvimage-iff*: $\langle a,b\rangle \in rvimage(A,f,r) \longleftrightarrow \langle f`a,f`b\rangle{:}\ r \wedge a \in A \wedge b \in A$
⟨*proof*⟩

20.3.2 Type checking

lemma *rvimage-type*: $rvimage(A,f,r) \subseteq A{*}A$
⟨*proof*⟩

lemmas *field-rvimage* = *rvimage-type* [*THEN field-rel-subset*]

lemma *rvimage-converse*: $rvimage(A,f,\ converse(r)) = converse(rvimage(A,f,r))$
⟨*proof*⟩

20.3.3 Partial Ordering Properties

lemma *irrefl-rvimage*:

$[\![f \in inj(A,B);\ \ irrefl(B,r)]\!] \Longrightarrow irrefl(A,\ rvimage(A,f,r))$
⟨*proof*⟩

lemma *trans-on-rvimage*:
$[\![f \in inj(A,B);\ \ trans[B](r)]\!] \Longrightarrow trans[A](rvimage(A,f,r))$
⟨*proof*⟩

lemma *part-ord-rvimage*:
$[\![f \in inj(A,B);\ \ part\text{-}ord(B,r)]\!] \Longrightarrow part\text{-}ord(A,\ rvimage(A,f,r))$
⟨*proof*⟩

20.3.4 Linearity

lemma *linear-rvimage*:
$[\![f \in inj(A,B);\ \ linear(B,r)]\!] \Longrightarrow linear(A,rvimage(A,f,r))$
⟨*proof*⟩

lemma *tot-ord-rvimage*:
$[\![f \in inj(A,B);\ \ tot\text{-}ord(B,r)]\!] \Longrightarrow tot\text{-}ord(A,\ rvimage(A,f,r))$
⟨*proof*⟩

20.3.5 Well-foundedness

lemma *wf-rvimage* [*intro*!]: $wf(r) \Longrightarrow wf(rvimage(A,f,r))$
⟨*proof*⟩

But note that the combination of *wf-imp-wf-on* and *wf-rvimage* gives $wf(r) \Longrightarrow wf[C](rvimage(A,\ f,\ r))$

lemma *wf-on-rvimage*: $[\![f \in A \to B;\ \ wf[B](r)]\!] \Longrightarrow wf[A](rvimage(A,f,r))$
⟨*proof*⟩

lemma *well-ord-rvimage*:
$[\![f \in inj(A,B);\ \ well\text{-}ord(B,r)]\!] \Longrightarrow well\text{-}ord(A,\ rvimage(A,f,r))$
⟨*proof*⟩

lemma *ord-iso-rvimage*:
$f \in bij(A,B) \Longrightarrow f \in ord\text{-}iso(A,\ rvimage(A,f,s),\ B,\ s)$
⟨*proof*⟩

lemma *ord-iso-rvimage-eq*:
$f \in ord\text{-}iso(A,r,\ B,s) \Longrightarrow rvimage(A,f,s) = r \cap A*A$
⟨*proof*⟩

20.4 Every well-founded relation is a subset of some inverse image of an ordinal

lemma *wf-rvimage-Ord*: $Ord(i) \Longrightarrow wf(rvimage(A,\ f,\ Memrel(i)))$
⟨*proof*⟩

definition
  wfrank :: $[i,i] \Rightarrow i$ **where**
    $wfrank(r,a) \equiv wfrec(r,\ a,\ \lambda x\ f.\ \bigcup y \in r-``\{x\}.\ succ(f`y))$

definition
  wftype :: $i \Rightarrow i$ **where**
    $wftype(r) \equiv \bigcup y \in range(r).\ succ(wfrank(r,y))$

lemma *wfrank*: $wf(r) \Longrightarrow wfrank(r,a) = (\bigcup y \in r-``\{a\}.\ succ(wfrank(r,y)))$
⟨*proof*⟩

lemma *Ord-wfrank*: $wf(r) \Longrightarrow Ord(wfrank(r,a))$
⟨*proof*⟩

lemma *wfrank-lt*: $[\![wf(r);\ \langle a,b\rangle \in r]\!] \Longrightarrow wfrank(r,a) < wfrank(r,b)$
⟨*proof*⟩

lemma *Ord-wftype*: $wf(r) \Longrightarrow Ord(wftype(r))$
⟨*proof*⟩

lemma *wftypeI*: $[\![wf(r);\ \ x \in field(r)]\!] \Longrightarrow wfrank(r,x) \in wftype(r)$
⟨*proof*⟩

lemma *wf-imp-subset-rvimage*:
    $[\![wf(r);\ r \subseteq A*A]\!] \Longrightarrow \exists\, i\ f.\ Ord(i) \wedge r \subseteq rvimage(A,\ f,\ Memrel(i))$
⟨*proof*⟩

theorem *wf-iff-subset-rvimage*:
  $relation(r) \Longrightarrow wf(r) \longleftrightarrow (\exists\, i\ f\ A.\ Ord(i) \wedge r \subseteq rvimage(A,\ f,\ Memrel(i)))$
⟨*proof*⟩

20.5 Other Results

lemma *wf-times*: $A \cap B = 0 \Longrightarrow wf(A*B)$
⟨*proof*⟩

Could also be used to prove *wf-radd*

lemma *wf-Un*:
    $[\![range(r) \cap domain(s) = 0;\ wf(r);\ \ wf(s)]\!] \Longrightarrow wf(r \cup s)$
⟨*proof*⟩

20.5.1 The Empty Relation

lemma *wf0*: $wf(0)$
⟨*proof*⟩

lemma *linear0*: *linear*(*0*,*0*)
⟨*proof*⟩

lemma *well-ord0*: *well-ord*(*0*,*0*)
⟨*proof*⟩

20.5.2 The "measure" relation is useful with wfrec

lemma *measure-eq-rvimage-Memrel*:
measure(*A*,*f*) = *rvimage*(*A*,*Lambda*(*A*,*f*),*Memrel*(*Collect*(*RepFun*(*A*,*f*),*Ord*)))
⟨*proof*⟩

lemma *wf-measure* [*iff*]: *wf*(*measure*(*A*,*f*))
⟨*proof*⟩

lemma *measure-iff* [*iff*]: ⟨*x*,*y*⟩ ∈ *measure*(*A*,*f*) ⟷ *x* ∈ *A* ∧ *y* ∈ *A* ∧ *f*(*x*)<*f*(*y*)
⟨*proof*⟩

lemma *linear-measure*:
assumes *Ordf*: ⋀*x*. *x* ∈ *A* ⟹ *Ord*(*f*(*x*))
and *inj*: ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *A*; *f*(*x*) = *f*(*y*)⟧ ⟹ *x*=*y*
shows *linear*(*A*, *measure*(*A*,*f*))
⟨*proof*⟩

lemma *wf-on-measure*: *wf*[*B*](*measure*(*A*,*f*))
⟨*proof*⟩

lemma *well-ord-measure*:
assumes *Ordf*: ⋀*x*. *x* ∈ *A* ⟹ *Ord*(*f*(*x*))
and *inj*: ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *A*; *f*(*x*) = *f*(*y*)⟧ ⟹ *x*=*y*
shows *well-ord*(*A*, *measure*(*A*,*f*))
⟨*proof*⟩

lemma *measure-type*: *measure*(*A*,*f*) ⊆ *A*∗*A*
⟨*proof*⟩

20.5.3 Well-foundedness of Unions

lemma *wf-on-Union*:
assumes *wfA*: *wf*[*A*](*r*)
and *wfB*: ⋀*a*. *a*∈*A* ⟹ *wf*[*B*(*a*)](*s*)
and *ok*: ⋀*a u v*. ⟦⟨*u*,*v*⟩ ∈ *s*; *v* ∈ *B*(*a*); *a* ∈ *A*⟧
⟹ (∃ *a*′∈*A*. ⟨*a*′,*a*⟩ ∈ *r* ∧ *u* ∈ *B*(*a*′)) | *u* ∈ *B*(*a*)
shows *wf*[⋃ *a*∈*A*. *B*(*a*)](*s*)
⟨*proof*⟩

20.5.4 Bijections involving Powersets

lemma *Pow-sum-bij*:
(λ*Z* ∈ *Pow*(*A*+*B*). ⟨{*x* ∈ *A*. *Inl*(*x*) ∈ *Z*}, {*y* ∈ *B*. *Inr*(*y*) ∈ *Z*}⟩)

$\in$ *bij*(*Pow*(*A*+*B*), *Pow*(*A*)∗*Pow*(*B*))
⟨*proof*⟩

As a special case, we have $bij(Pow(A \times B), A \to Pow(B))$

lemma *Pow-Sigma-bij*:
(λ*r* ∈ *Pow*(*Sigma*(*A*,*B*)). λ*x* ∈ *A*. *r*‘‘{*x*})
∈ *bij*(*Pow*(*Sigma*(*A*,*B*)), ∏ *x* ∈ *A*. *Pow*(*B*(*x*)))
⟨*proof*⟩

end

21 Order Types and Ordinal Arithmetic

theory *OrderType* **imports** *OrderArith OrdQuant Nat* **begin**

The order type of a well-ordering is the least ordinal isomorphic to it. Ordinal arithmetic is traditionally defined in terms of order types, as it is here. But a definition by transfinite recursion would be much simpler!

definition
ordermap :: [*i*,*i*]⇒*i* **where**
ordermap(*A*,*r*) ≡ λ*x*∈*A*. *wfrec*[*A*](*r*, *x*, λ*x f*. *f* ‘‘ *pred*(*A*,*x*,*r*))

definition
ordertype :: [*i*,*i*]⇒*i* **where**
ordertype(*A*,*r*) ≡ *ordermap*(*A*,*r*)‘‘*A*

definition

Ord-alt :: *i* ⇒ *o* **where**
Ord-alt(*X*) ≡ *well-ord*(*X*, *Memrel*(*X*)) ∧ (∀ *u*∈*X*. *u*=*pred*(*X*, *u*, *Memrel*(*X*)))

definition

ordify :: *i*⇒*i* **where**
ordify(*x*) ≡ *if Ord*(*x*) *then x else 0*

definition

omult :: [*i*,*i*]⇒*i* (**infixl** ‹∗∗› *70*) **where**
i ∗∗ *j* ≡ *ordertype*(*j*∗*i*, *rmult*(*j*,*Memrel*(*j*),*i*,*Memrel*(*i*)))

definition

raw-oadd :: [*i*,*i*]⇒*i* **where**
raw-oadd(*i*,*j*) ≡ *ordertype*(*i*+*j*, *radd*(*i*,*Memrel*(*i*),*j*,*Memrel*(*j*)))

definition
oadd :: [*i*,*i*]⇒*i* (**infixl** ‹++› *65*) **where**

i ++ $j \equiv$ *raw-oadd*(*ordify*(i),*ordify*(j))

definition

odiff :: [i,i]⇒i (**infixl** ‹−−› 65) **where**
i −− $j \equiv$ *ordertype*(i−j, *Memrel*(i))

21.1 Proofs needing the combination of Ordinal.thy and Order.thy

lemma *le-well-ord-Memrel*: $j \leq i \Longrightarrow$ *well-ord*(j, *Memrel*(i))
⟨*proof*⟩

lemmas *well-ord-Memrel* = *le-refl* [*THEN le-well-ord-Memrel*]

lemma *lt-pred-Memrel*:
j<$i \Longrightarrow$ *pred*(i, j, *Memrel*(i)) = j
⟨*proof*⟩

lemma *pred-Memrel*:
$x \in A \Longrightarrow$ *pred*(A, x, *Memrel*(A)) = $A \cap x$
⟨*proof*⟩

lemma *Ord-iso-implies-eq-lemma*:
⟦j<i; $f \in$ *ord-iso*(i,*Memrel*(i),j,*Memrel*(j))⟧ ⟹ R
⟨*proof*⟩

lemma *Ord-iso-implies-eq*:
⟦*Ord*(i); *Ord*(j); $f \in$ *ord-iso*(i,*Memrel*(i),j,*Memrel*(j))⟧
⟹ i=j
⟨*proof*⟩

21.2 Ordermap and ordertype

lemma *ordermap-type*:
ordermap(A,r) ∈ A −> *ordertype*(A,r)
⟨*proof*⟩

21.2.1 Unfolding of ordermap

lemma *ordermap-eq-image*:
⟦*wf*[A](r); $x \in A$⟧
⟹ *ordermap*(A,r) ‘ x = *ordermap*(A,r) ‘‘ *pred*(A,x,r)
⟨*proof*⟩

lemma *ordermap-pred-unfold*:

$[\![wf[A](r);\ \ x \in A]\!]$
$\Longrightarrow ordermap(A,r)\ `\ x = \{ordermap(A,r)`y\ .\ y \in pred(A,x,r)\}$
⟨*proof*⟩

lemmas *ordermap-unfold* = *ordermap-pred-unfold* [*simplified pred-def*]

21.2.2 Showing that ordermap, ordertype yield ordinals

lemma *Ord-ordermap*:
$[\![well\text{-}ord(A,r);\ \ x \in A]\!] \Longrightarrow Ord(ordermap(A,r)\ `\ x)$
⟨*proof*⟩

lemma *Ord-ordertype*:
$well\text{-}ord(A,r) \Longrightarrow Ord(ordertype(A,r))$
⟨*proof*⟩

21.2.3 ordermap preserves the orderings in both directions

lemma *ordermap-mono*:
$[\![\langle w,x\rangle : r;\ \ wf[A](r);\ \ w \in A;\ x \in A]\!]$
$\Longrightarrow ordermap(A,r)`w \in ordermap(A,r)`x$
⟨*proof*⟩

lemma *converse-ordermap-mono*:
$[\![ordermap(A,r)`w \in ordermap(A,r)`x;\ \ well\text{-}ord(A,r);\ w \in A;\ x \in A]\!]$
$\Longrightarrow \langle w,x\rangle : r$
⟨*proof*⟩

lemma *ordermap-surj*: $ordermap(A,\ r) \in surj(A,\ ordertype(A,\ r))$
⟨*proof*⟩

lemma *ordermap-bij*:
$well\text{-}ord(A,r) \Longrightarrow ordermap(A,r) \in bij(A,\ ordertype(A,r))$
⟨*proof*⟩

21.2.4 Isomorphisms involving ordertype

lemma *ordertype-ord-iso*:
$well\text{-}ord(A,r)$
$\Longrightarrow ordermap(A,r) \in ord\text{-}iso(A,r,\ ordertype(A,r),\ Memrel(ordertype(A,r)))$
⟨*proof*⟩

lemma *ordertype-eq*:
$[\![f \in ord\text{-}iso(A,r,B,s);\ \ well\text{-}ord(B,s)]\!]$
$\Longrightarrow ordertype(A,r) = ordertype(B,s)$
⟨*proof*⟩

lemma *ordertype-eq-imp-ord-iso*:

$[\![ordertype(A,r) = ordertype(B,s);\ well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!]$
$\Longrightarrow \exists f.\ f \in ord\text{-}iso(A,r,B,s)$
⟨*proof*⟩

21.2.5 Basic equalities for ordertype

lemma *le-ordertype-Memrel*: $j \leq i \Longrightarrow ordertype(j,Memrel(i)) = j$
⟨*proof*⟩

lemmas *ordertype-Memrel* = *le-refl* [*THEN le-ordertype-Memrel*]

lemma *ordertype-0* [*simp*]: $ordertype(0,r) = 0$
⟨*proof*⟩

lemmas *bij-ordertype-vimage* = *ord-iso-rvimage* [*THEN ordertype-eq*]

21.2.6 A fundamental unfolding law for ordertype.

lemma *ordermap-pred-eq-ordermap*:
$[\![well\text{-}ord(A,r);\ \ y \in A;\ \ z \in pred(A,y,r)]\!]$
$\Longrightarrow ordermap(pred(A,y,r),\ r)\ `z = ordermap(A,\ r)\ `z$
⟨*proof*⟩

lemma *ordertype-unfold*:
$ordertype(A,r) = \{ordermap(A,r)`y\ .\ y \in A\}$
⟨*proof*⟩

Theorems by Krzysztof Grabczewski; proofs simplified by lcp

lemma *ordertype-pred-subset*: $[\![well\text{-}ord(A,r);\ \ x \in A]\!] \Longrightarrow$
$ordertype(pred(A,x,r),r) \subseteq ordertype(A,r)$
⟨*proof*⟩

lemma *ordertype-pred-lt*:
$[\![well\text{-}ord(A,r);\ \ x \in A]\!]$
$\Longrightarrow ordertype(pred(A,x,r),r) < ordertype(A,r)$
⟨*proof*⟩

lemma *ordertype-pred-unfold*:
$well\text{-}ord(A,r)$
$\Longrightarrow ordertype(A,r) = \{ordertype(pred(A,x,r),r).\ x \in A\}$
⟨*proof*⟩

21.3 Alternative definition of ordinal

lemma *Ord-is-Ord-alt*: $Ord(i) \Longrightarrow Ord\text{-}alt(i)$
⟨*proof*⟩

lemma *Ord-alt-is-Ord*:
$\quad$ *Ord-alt*(i) $\Longrightarrow$ *Ord*(i)
⟨*proof*⟩

21.4 Ordinal Addition

21.4.1 Order Type calculations for radd

Addition with 0

lemma *bij-sum-0*: ($\lambda z{\in}A{+}0.$ *case*($\lambda x.\ x$, $\lambda y.\ y$, z)) $\in$ *bij*($A{+}0$, A)
⟨*proof*⟩

lemma *ordertype-sum-0-eq*:
$\quad$ *well-ord*(A,r) $\Longrightarrow$ *ordertype*($A{+}0$, *radd*(A,r,0,s)) = *ordertype*(A,r)
⟨*proof*⟩

lemma *bij-0-sum*: ($\lambda z{\in}0{+}A.$ *case*($\lambda x.\ x$, $\lambda y.\ y$, z)) $\in$ *bij*($0{+}A$, A)
⟨*proof*⟩

lemma *ordertype-0-sum-eq*:
$\quad$ *well-ord*(A,r) $\Longrightarrow$ *ordertype*($0{+}A$, *radd*(0,s,A,r)) = *ordertype*(A,r)
⟨*proof*⟩

Initial segments of radd. Statements by Grabczewski

lemma *pred-Inl-bij*:
$a \in A \Longrightarrow (\lambda x{\in}\mathit{pred}(A,a,r).\ \mathit{Inl}(x))$
$\quad\in \mathit{bij}(\mathit{pred}(A,a,r),\ \mathit{pred}(A{+}B,\ \mathit{Inl}(a),\ \mathit{radd}(A,r,B,s)))$
⟨*proof*⟩

lemma *ordertype-pred-Inl-eq*:
$\quad[\![a \in A;\ \ \mathit{well\text{-}ord}(A,r)]\!]$
$\quad\Longrightarrow \mathit{ordertype}(\mathit{pred}(A{+}B,\ \mathit{Inl}(a),\ \mathit{radd}(A,r,B,s)),\ \mathit{radd}(A,r,B,s)) =$
$\quad\quad \mathit{ordertype}(\mathit{pred}(A,a,r),\ r)$
⟨*proof*⟩

lemma *pred-Inr-bij*:
$b \in B \Longrightarrow$
$\quad \mathit{id}(A{+}\mathit{pred}(B,b,s))$
$\quad \in \mathit{bij}(A{+}\mathit{pred}(B,b,s),\ \mathit{pred}(A{+}B,\ \mathit{Inr}(b),\ \mathit{radd}(A,r,B,s)))$
⟨*proof*⟩

lemma *ordertype-pred-Inr-eq*:
$\quad[\![b \in B;\ \ \mathit{well\text{-}ord}(A,r);\ \ \mathit{well\text{-}ord}(B,s)]\!]$
$\quad\Longrightarrow \mathit{ordertype}(\mathit{pred}(A{+}B,\ \mathit{Inr}(b),\ \mathit{radd}(A,r,B,s)),\ \mathit{radd}(A,r,B,s)) =$
$\quad\quad \mathit{ordertype}(A{+}\mathit{pred}(B,b,s),\ \mathit{radd}(A,r,\mathit{pred}(B,b,s),s))$
⟨*proof*⟩

21.4.2 ordify: trivial coercion to an ordinal

lemma *Ord-ordify* [*iff*, *TC*]: *Ord*(*ordify*(*x*))
⟨*proof*⟩

lemma *ordify-idem* [*simp*]: *ordify*(*ordify*(*x*)) = *ordify*(*x*)
⟨*proof*⟩

21.4.3 Basic laws for ordinal addition

lemma *Ord-raw-oadd*: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ *Ord*(*raw-oadd*(*i*,*j*))
⟨*proof*⟩

lemma *Ord-oadd* [*iff*,*TC*]: *Ord*(*i*++*j*)
⟨*proof*⟩

Ordinal addition with zero

lemma *raw-oadd-0*: *Ord*(*i*) ⟹ *raw-oadd*(*i*,*0*) = *i*
⟨*proof*⟩

lemma *oadd-0* [*simp*]: *Ord*(*i*) ⟹ *i*++*0* = *i*
⟨*proof*⟩

lemma *raw-oadd-0-left*: *Ord*(*i*) ⟹ *raw-oadd*(*0*,*i*) = *i*
⟨*proof*⟩

lemma *oadd-0-left* [*simp*]: *Ord*(*i*) ⟹ *0*++*i* = *i*
⟨*proof*⟩

lemma *oadd-eq-if-raw-oadd*:
i++*j* = (*if Ord*(*i*) *then* (*if Ord*(*j*) *then raw-oadd*(*i*,*j*) *else i*)
else (*if Ord*(*j*) *then j else 0*))
⟨*proof*⟩

lemma *raw-oadd-eq-oadd*: ⟦*Ord*(*i*); *Ord*(*j*)⟧ ⟹ *raw-oadd*(*i*,*j*) = *i*++*j*
⟨*proof*⟩

lemma *lt-oadd1*: *k*<*i* ⟹ *k* < *i*++*j*
⟨*proof*⟩

lemma *oadd-le-self*: *Ord*(*i*) ⟹ *i* ≤ *i*++*j*
⟨*proof*⟩

Various other results

lemma *id-ord-iso-Memrel*: $A<=B \Longrightarrow id(A) \in ord\text{-}iso(A,\ Memrel(A),\ A,\ Memrel(B))$
⟨*proof*⟩

lemma *subset-ord-iso-Memrel*:
$[\![f \in ord\text{-}iso(A,Memrel(B),C,r);\ A<=B]\!] \Longrightarrow f \in ord\text{-}iso(A,Memrel(A),C,r)$
⟨*proof*⟩

lemma *restrict-ord-iso*:
$[\![f \in ord\text{-}iso(i,\ Memrel(i),\ Order.pred(A,a,r),\ r);\ \ a \in A;\ j < i;$
$\ trans[A](r)]\!]$
$\Longrightarrow restrict(f,j) \in ord\text{-}iso(j,\ Memrel(j),\ Order.pred(A,f`j,r),\ r)$
⟨*proof*⟩

lemma *restrict-ord-iso2*:
$[\![f \in ord\text{-}iso(Order.pred(A,a,r),\ r,\ i,\ Memrel(i));\ \ a \in A;$
$\ j < i;\ trans[A](r)]\!]$
$\Longrightarrow converse(restrict(converse(f),\ j))$
$\in ord\text{-}iso(Order.pred(A,\ converse(f)`j,\ r),\ r,\ j,\ Memrel(j))$
⟨*proof*⟩

lemma *ordertype-sum-Memrel*:
$[\![well\text{-}ord(A,r);\ \ k<j]\!]$
$\Longrightarrow ordertype(A+k,\ radd(A,\ r,\ k,\ Memrel(j))) =$
$ordertype(A+k,\ radd(A,\ r,\ k,\ Memrel(k)))$
⟨*proof*⟩

lemma *oadd-lt-mono2*: $k<j \Longrightarrow i{+}{+}k < i{+}{+}j$
⟨*proof*⟩

lemma *oadd-lt-cancel2*: $[\![i{+}{+}j < i{+}{+}k;\ \ Ord(j)]\!] \Longrightarrow j<k$
⟨*proof*⟩

lemma *oadd-lt-iff2*: $Ord(j) \Longrightarrow i{+}{+}j < i{+}{+}k \longleftrightarrow j<k$
⟨*proof*⟩

lemma *oadd-inject*: $[\![i{+}{+}j = i{+}{+}k;\ \ Ord(j);\ Ord(k)]\!] \Longrightarrow j=k$
⟨*proof*⟩

lemma *lt-oadd-disj*: $k < i{+}{+}j \Longrightarrow k<i \mid (\exists\, l{\in}j.\ k = i{+}{+}l\ )$
⟨*proof*⟩

21.4.4 Ordinal addition with successor – via associativity!

lemma *oadd-assoc*: $(i{+}{+}j){+}{+}k = i{+}{+}(j{+}{+}k)$
⟨*proof*⟩

lemma *oadd-unfold*: $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow i{+}{+}j = i \cup (\bigcup k{\in}j.\ \{i{+}{+}k\})$
⟨*proof*⟩

lemma *oadd-1*: $Ord(i) \Longrightarrow i{+}{+}1 = succ(i)$
⟨*proof*⟩

lemma *oadd-succ* [*simp*]: $Ord(j) \Longrightarrow i{+}{+}succ(j) = succ(i{+}{+}j)$
⟨*proof*⟩

Ordinal addition with limit ordinals

lemma *oadd-UN*:
$[\![\bigwedge x.\ x \in A \Longrightarrow Ord(j(x));\ \ a \in A]\!]$
$\Longrightarrow i +{+} (\bigcup x{\in}A.\ j(x)) = (\bigcup x{\in}A.\ i{+}{+}j(x))$
⟨*proof*⟩

lemma *oadd-Limit*: $Limit(j) \Longrightarrow i{+}{+}j = (\bigcup k{\in}j.\ i{+}{+}k)$
⟨*proof*⟩

lemma *oadd-eq-0-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow (i +{+} j) = 0 \longleftrightarrow i{=}0 \wedge j{=}0$
⟨*proof*⟩

lemma *oadd-eq-lt-iff*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow 0 < (i +{+} j) \longleftrightarrow 0{<}i \mid 0{<}j$
⟨*proof*⟩

lemma *oadd-LimitI*: $[\![Ord(i);\ Limit(j)]\!] \Longrightarrow Limit(i +{+} j)$
⟨*proof*⟩

Order/monotonicity properties of ordinal addition

lemma *oadd-le-self2*: $Ord(i) \Longrightarrow i \leq j{+}{+}i$
⟨*proof*⟩

lemma *oadd-le-mono1*: $k \leq j \Longrightarrow k{+}{+}i \leq j{+}{+}i$
⟨*proof*⟩

lemma *oadd-lt-mono*: $[\![i' \leq i;\ \ j'{<}j]\!] \Longrightarrow i'{+}{+}j' < i{+}{+}j$
⟨*proof*⟩

lemma *oadd-le-mono*: $[\![i' \leq i;\ \ j' \leq j]\!] \Longrightarrow i'{+}{+}j' \leq i{+}{+}j$
⟨*proof*⟩

lemma *oadd-le-iff2*: $[\![Ord(j);\ Ord(k)]\!] \Longrightarrow i{+}{+}j \leq i{+}{+}k \longleftrightarrow j \leq k$
⟨*proof*⟩

lemma *oadd-lt-self*: $[\![Ord(i);\ \ 0{<}j]\!] \Longrightarrow i < i{+}{+}j$
⟨*proof*⟩

Every ordinal is exceeded by some limit ordinal.

lemma *Ord-imp-greater-Limit*: $Ord(i) \Longrightarrow \exists\, k.\ i{<}k \wedge Limit(k)$
⟨*proof*⟩

lemma *Ord2-imp-greater-Limit*: $[\![Ord(i);\ Ord(j)]\!] \Longrightarrow \exists\, k.\ i{<}k \wedge j{<}k \wedge Limit(k)$

⟨*proof*⟩

21.5 Ordinal Subtraction

The difference is *ordertype*($j - i$, *Memrel*(j)). It's probably simpler to define the difference recursively!

lemma *bij-sum-Diff*:
$A{<}{=}B \Longrightarrow (\lambda y{\in}B.\ \mathit{if}(y \in A,\ \mathit{Inl}(y),\ \mathit{Inr}(y))) \in \mathit{bij}(B,\ A{+}(B{-}A))$
⟨*proof*⟩

lemma *ordertype-sum-Diff*:
$i \le j \Longrightarrow$
$\mathit{ordertype}(i{+}(j{-}i),\ \mathit{radd}(i,\mathit{Memrel}(j),j{-}i,\mathit{Memrel}(j))) =$
$\mathit{ordertype}(j,\ \mathit{Memrel}(j))$
⟨*proof*⟩

lemma *Ord-odiff* [*simp*, *TC*]:
$[\![\mathit{Ord}(i);\ \ \mathit{Ord}(j)]\!] \Longrightarrow \mathit{Ord}(i{-}{-}j)$
⟨*proof*⟩

lemma *raw-oadd-ordertype-Diff*:
$i \le j$
$\Longrightarrow \mathit{raw\text{-}oadd}(i,j{-}{-}i) = \mathit{ordertype}(i{+}(j{-}i),\ \mathit{radd}(i,\mathit{Memrel}(j),j{-}i,\mathit{Memrel}(j)))$
⟨*proof*⟩

lemma *oadd-odiff-inverse*: $i \le j \Longrightarrow i +{+} (j{-}{-}i) = j$
⟨*proof*⟩

lemma *odiff-oadd-inverse*: $[\![\mathit{Ord}(i);\ \mathit{Ord}(j)]\!] \Longrightarrow (i{+}{+}j) -{-} i = j$
⟨*proof*⟩

lemma *odiff-lt-mono2*: $[\![i{<}j;\ \ k \le i]\!] \Longrightarrow i{-}{-}k < j{-}{-}k$
⟨*proof*⟩

21.6 Ordinal Multiplication

lemma *Ord-omult* [*simp*, *TC*]:
$[\![\mathit{Ord}(i);\ \ \mathit{Ord}(j)]\!] \Longrightarrow \mathit{Ord}(i{*}{*}j)$
⟨*proof*⟩

21.6.1 A useful unfolding law

lemma *pred-Pair-eq*:
$[\![a \in A;\ \ b \in B]\!] \Longrightarrow \mathit{pred}(A{*}B,\ \langle a,b\rangle,\ \mathit{rmult}(A,r,B,s)) =$
$\mathit{pred}(A,a,r){*}B \cup (\{a\} * \mathit{pred}(B,b,s))$
⟨*proof*⟩

lemma *ordertype-pred-Pair-eq*:
$[\![a \in A;\ \ b \in B;\ \ well\text{-}ord(A,r);\ \ well\text{-}ord(B,s)]\!] \Longrightarrow$
$ordertype(pred(A*B,\ \langle a,b\rangle,\ rmult(A,r,B,s)),\ rmult(A,r,B,s)) =$
$ordertype(pred(A,a,r)*B\ +\ pred(B,b,s),$
$radd(A*B,\ rmult(A,r,B,s),\ B,\ s))$
⟨*proof*⟩

lemma *ordertype-pred-Pair-lemma*:
$[\![i'<i;\ \ j'<j]\!]$
$\Longrightarrow ordertype(pred(i*j,\ <i',j'>,\ rmult(i,Memrel(i),j,Memrel(j))),$
$rmult(i,Memrel(i),j,Memrel(j))) =$
$raw\text{-}oadd\ (j**i',\ j')$
⟨*proof*⟩

lemma *lt-omult*:
$[\![Ord(i);\ \ Ord(j);\ \ k<j**i]\!]$
$\Longrightarrow \exists j'\ i'.\ k = j**i' +\!\!+ j' \wedge j'<j \wedge i'<i$
⟨*proof*⟩

lemma *omult-oadd-lt*:
$[\![j'<j;\ \ i'<i]\!] \Longrightarrow j**i' +\!\!+ j'\ <\ j**i$
⟨*proof*⟩

lemma *omult-unfold*:
$[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow j**i = (\bigcup j'\in j.\ \bigcup i'\in i.\ \{j**i' +\!\!+ j'\})$
⟨*proof*⟩

21.6.2 Basic laws for ordinal multiplication

Ordinal multiplication by zero

lemma *omult-0* [*simp*]: $i**0 = 0$
⟨*proof*⟩

lemma *omult-0-left* [*simp*]: $0**i = 0$
⟨*proof*⟩

Ordinal multiplication by 1

lemma *omult-1* [*simp*]: $Ord(i) \Longrightarrow i**1 = i$
⟨*proof*⟩

lemma *omult-1-left* [*simp*]: $Ord(i) \Longrightarrow 1**i = i$
⟨*proof*⟩

Distributive law for ordinal multiplication and addition

lemma *oadd-omult-distrib*:
$[\![Ord(i);\ \ Ord(j);\ \ Ord(k)]\!] \Longrightarrow i**(j+\!\!+k) = (i**j)+\!\!+(i**k)$
⟨*proof*⟩

lemma *omult-succ*: $[\![Ord(i);\ \ Ord(j)]\!] \Longrightarrow i{**}succ(j) = (i{**}j){+}{+}i$
⟨*proof*⟩

Associative law

lemma *omult-assoc*:
$[\![Ord(i);\ \ Ord(j);\ \ Ord(k)]\!] \Longrightarrow (i{**}j){**}k = i{**}(j{**}k)$
⟨*proof*⟩

Ordinal multiplication with limit ordinals

lemma *omult-UN*:
$[\![Ord(i);\ \bigwedge x.\ x \in A \Longrightarrow Ord(j(x))]\!]$
$\Longrightarrow i\ **\ (\bigcup x{\in}A.\ j(x)) = (\bigcup x{\in}A.\ i{**}j(x))$
⟨*proof*⟩

lemma *omult-Limit*: $[\![Ord(i);\ \ Limit(j)]\!] \Longrightarrow i{**}j = (\bigcup k{\in}j.\ i{**}k)$
⟨*proof*⟩

21.6.3 Ordering/monotonicity properties of ordinal multiplication

lemma *lt-omult1*: $[\![k{<}i;\ \ 0{<}j]\!] \Longrightarrow k < i{**}j$
⟨*proof*⟩

lemma *omult-le-self*: $[\![Ord(i);\ \ 0{<}j]\!] \Longrightarrow i \le i{**}j$
⟨*proof*⟩

lemma *omult-le-mono1*:
assumes *kj*: $k \le j$ **and** *i*: $Ord(i)$ **shows** $k{**}i \le j{**}i$
⟨*proof*⟩

lemma *omult-lt-mono2*: $[\![k{<}j;\ \ 0{<}i]\!] \Longrightarrow i{**}k < i{**}j$
⟨*proof*⟩

lemma *omult-le-mono2*: $[\![k \le j;\ \ Ord(i)]\!] \Longrightarrow i{**}k \le i{**}j$
⟨*proof*⟩

lemma *omult-le-mono*: $[\![i' \le i;\ \ j' \le j]\!] \Longrightarrow i'{**}j' \le i{**}j$
⟨*proof*⟩

lemma *omult-lt-mono*: $[\![i' \le i;\ \ j'{<}j;\ \ 0{<}i]\!] \Longrightarrow i'{**}j' < i{**}j$
⟨*proof*⟩

lemma *omult-le-self2*:
assumes *i*: $Ord(i)$ **and** *j*: $0{<}j$ **shows** $i \le j{**}i$
⟨*proof*⟩

Further properties of ordinal multiplication

lemma *omult-inject*: $[\![i{**}j = i{**}k;\ \ 0{<}i;\ \ Ord(j);\ \ Ord(k)]\!] \Longrightarrow j{=}k$
⟨*proof*⟩

21.7 The Relation *Lt*

lemma *wf-Lt*: *wf*(*Lt*)
⟨*proof*⟩

lemma *irrefl-Lt*: *irrefl*(*A*,*Lt*)
⟨*proof*⟩

lemma *trans-Lt*: *trans*[*A*](*Lt*)
⟨*proof*⟩

lemma *part-ord-Lt*: *part-ord*(*A*,*Lt*)
⟨*proof*⟩

lemma *linear-Lt*: *linear*(*nat*,*Lt*)
⟨*proof*⟩

lemma *tot-ord-Lt*: *tot-ord*(*nat*,*Lt*)
⟨*proof*⟩

lemma *well-ord-Lt*: *well-ord*(*nat*,*Lt*)
⟨*proof*⟩

end

22 Finite Powerset Operator and Finite Function Space

theory *Finite* **imports** *Inductive Epsilon Nat* **begin**

rep-datatype
  elimination *natE*
  induction *nat-induct*
  case-eqns *nat-case-0 nat-case-succ*
  recursor-eqns *recursor-0 recursor-succ*

consts
  Fin :: *i*⇒*i*
  FiniteFun :: [*i*,*i*]⇒*i* (‹(- −||>/ -)› [*61*, *60*] *60*)

inductive
  domains *Fin*(*A*) ⊆ *Pow*(*A*)
  intros
    emptyI: *0* ∈ *Fin*(*A*)
    consI: ⟦*a* ∈ *A*; *b* ∈ *Fin*(*A*)⟧ ⟹ *cons*(*a*,*b*) ∈ *Fin*(*A*)
  type-intros *empty-subsetI cons-subsetI PowI*
  type-elims *PowD* [*elim-format*]

inductive
  domains $FiniteFun(A,B) \subseteq Fin(A*B)$
  intros
    $emptyI$: $0 \in A \mathrel{-||>} B$
    $consI$: $[\![a \in A;\ b \in B;\ h \in A \mathrel{-||>} B;\ a \notin domain(h)]\!]$
      $\Longrightarrow cons(\langle a,b\rangle,h) \in A \mathrel{-||>} B$
  type-intros *Fin.intros*

22.1 Finite Powerset Operator

lemma *Fin-mono*: $A<=B \Longrightarrow Fin(A) \subseteq Fin(B)$
⟨*proof*⟩

lemmas $FinD = Fin.dom\text{-}subset\ [THEN\ subsetD,\ THEN\ PowD]$

lemma *Fin-induct* [*case-names 0 cons*, *induct set*: *Fin*]:
  $[\![b \in Fin(A);$
    $P(0);$
    $\bigwedge x\ y.\ [\![x \in A;\ y \in Fin(A);\ x\notin y;\ P(y)]\!] \Longrightarrow P(cons(x,y))$
$]\!] \Longrightarrow P(b)$
⟨*proof*⟩

declare *Fin.intros* [*simp*]

lemma *Fin-0*: $Fin(0) = \{0\}$
⟨*proof*⟩

lemma *Fin-UnI* [*simp*]: $[\![b \in Fin(A);\ c \in Fin(A)]\!] \Longrightarrow b \cup c \in Fin(A)$
⟨*proof*⟩

lemma *Fin-UnionI*: $C \in Fin(Fin(A)) \Longrightarrow \bigcup(C) \in Fin(A)$
⟨*proof*⟩

lemma *Fin-subset-lemma* [*rule-format*]: $b \in Fin(A) \Longrightarrow \forall z.\ z<=b \longrightarrow z \in Fin(A)$
⟨*proof*⟩

lemma *Fin-subset*: $[\![c<=b;\ b \in Fin(A)]\!] \Longrightarrow c \in Fin(A)$
⟨*proof*⟩

lemma *Fin-IntI1* [*intro*,*simp*]: $b \in Fin(A) \Longrightarrow b \cap c \in Fin(A)$
⟨*proof*⟩

lemma *Fin-IntI2* [*intro*,*simp*]: $c \in Fin(A) \Longrightarrow b \cap c \in Fin(A)$
⟨*proof*⟩

lemma *Fin-0-induct-lemma* [*rule-format*]:
$[\![c \in Fin(A);\ \ b \in Fin(A);\ P(b);$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in Fin(A);\ \ x \in y;\ \ P(y)]\!] \Longrightarrow P(y-\{x\})$
$]\!] \Longrightarrow c<=b \longrightarrow P(b-c)$
⟨*proof*⟩

lemma *Fin-0-induct*:
$[\![b \in Fin(A);$
$P(b);$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in Fin(A);\ \ x \in y;\ \ P(y)]\!] \Longrightarrow P(y-\{x\})$
$]\!] \Longrightarrow P(0)$
⟨*proof*⟩

lemma *nat-fun-subset-Fin*: $n \in nat \Longrightarrow n->A \subseteq Fin(nat*A)$
⟨*proof*⟩

22.2 Finite Function Space

lemma *FiniteFun-mono*:
$[\![A<=C;\ \ B<=D]\!] \Longrightarrow A\ -||>\ B\ \subseteq\ C\ -||>\ D$
⟨*proof*⟩

lemma *FiniteFun-mono1*: $A<=B \Longrightarrow A\ -||>\ A\ \subseteq\ B\ -||>\ B$
⟨*proof*⟩

lemma *FiniteFun-is-fun*: $h \in A\ -||>B \Longrightarrow h \in domain(h)\ -> B$
⟨*proof*⟩

lemma *FiniteFun-domain-Fin*: $h \in A\ -||>B \Longrightarrow domain(h) \in Fin(A)$
⟨*proof*⟩

lemmas *FiniteFun-apply-type* = *FiniteFun-is-fun* [*THEN apply-type*]

lemma *FiniteFun-subset-lemma* [*rule-format*]:
$b \in A-||>B \Longrightarrow \forall z.\ z<=b \longrightarrow z \in A-||>B$
⟨*proof*⟩

lemma *FiniteFun-subset*: $[\![c<=b;\ \ b \in A-||>B]\!] \Longrightarrow c \in A-||>B$
⟨*proof*⟩

lemma *fun-FiniteFunI* [*rule-format*]: $A \in Fin(X) \Longrightarrow \forall f.\ f \in A->B \longrightarrow f \in A-||>B$
⟨*proof*⟩

lemma *lam-FiniteFun*: $A \in Fin(X) \Longrightarrow (\lambda x{\in}A.\ b(x)) \in A\ -||>\ \{b(x).\ x \in A\}$
⟨*proof*⟩

lemma *FiniteFun-Collect-iff*:
$f \in FiniteFun(A, \{y \in B.\ P(y)\})$
$\longleftrightarrow f \in FiniteFun(A,B) \wedge (\forall x{\in}domain(f).\ P(f`x))$
⟨*proof*⟩

22.3 The Contents of a Singleton Set

definition
contents :: $i \Rightarrow i$ **where**
$contents(X) \equiv THE\ x.\ X = \{x\}$

lemma *contents-eq* [*simp*]: $contents\ (\{x\}) = x$
⟨*proof*⟩

end

23 Cardinal Numbers Without the Axiom of Choice

theory *Cardinal* **imports** *OrderType Finite Nat Sum* **begin**

definition

Least :: $(i \Rightarrow o) \Rightarrow i$ (**binder** $\langle\mu\ \rangle$ *10*) **where**
$Least(P) \equiv THE\ i.\ Ord(i) \wedge P(i) \wedge (\forall j.\ j{<}i \longrightarrow \neg P(j))$

definition
eqpoll :: $[i,i] \Rightarrow o$ (**infixl** $\langle\approx\rangle$ *50*) **where**
$A \approx B \equiv \exists f.\ f \in bij(A,B)$

definition
lepoll :: $[i,i] \Rightarrow o$ (**infixl** $\langle\lesssim\rangle$ *50*) **where**
$A \lesssim B \equiv \exists f.\ f \in inj(A,B)$

definition
lesspoll :: $[i,i] \Rightarrow o$ (**infixl** $\langle\prec\rangle$ *50*) **where**
$A \prec B \equiv A \lesssim B \wedge \neg(A \approx B)$

definition
cardinal :: $i \Rightarrow i$ ($\langle|\text{-}|\rangle$) **where**
$|A| \equiv (\mu\ i.\ i \approx A)$

definition
Finite :: $i \Rightarrow o$ **where**
$Finite(A) \equiv \exists\, n{\in}nat.\ A \approx n$

definition
Card :: $i \Rightarrow o$ **where**
$Card(i) \equiv (i = |i|)$

23.1 The Schroeder-Bernstein Theorem

See Davey and Priestly, page 106

lemma *decomp-bnd-mono*: $bnd\text{-}mono(X,\ \lambda W.\ X - g``(Y - f``W))$
⟨*proof*⟩

lemma *Banach-last-equation*:
$g \in Y{->}X$
$\Longrightarrow g``(Y - f`` lfp(X,\ \lambda W.\ X - g``(Y - f``W))) =$
$X - lfp(X,\ \lambda W.\ X - g``(Y - f``W))$
⟨*proof*⟩

lemma *decomposition*:
$[\![f \in X{->}Y;\ \ g \in Y{->}X]\!] \Longrightarrow$
$\exists\, XA\ XB\ YA\ YB.\ (XA \cap XB = 0) \wedge (XA \cup XB = X) \wedge$
$(YA \cap YB = 0) \wedge (YA \cup YB = Y) \wedge$
$f``XA{=}YA \wedge g``YB{=}XB$
⟨*proof*⟩

lemma *schroeder-bernstein*:
$[\![f \in inj(X,Y);\ \ g \in inj(Y,X)]\!] \Longrightarrow \exists\, h.\ h \in bij(X,Y)$
⟨*proof*⟩

lemma *bij-imp-eqpoll*: $f \in bij(A,B) \Longrightarrow A \approx B$
⟨*proof*⟩

lemmas *eqpoll-refl* $=$ *id-bij* [*THEN bij-imp-eqpoll*, *simp*]

lemma *eqpoll-sym*: $X \approx Y \Longrightarrow Y \approx X$
⟨*proof*⟩

lemma *eqpoll-trans* [*trans*]:
$[\![X \approx Y;\ \ Y \approx Z]\!] \Longrightarrow X \approx Z$
⟨*proof*⟩

lemma *subset-imp-lepoll*: $X <= Y \Longrightarrow X \lesssim Y$
⟨*proof*⟩

lemmas *lepoll-refl* = *subset-refl* [*THEN subset-imp-lepoll*, *simp*]

lemmas *le-imp-lepoll* = *le-imp-subset* [*THEN subset-imp-lepoll*]

lemma *eqpoll-imp-lepoll*: $X \approx Y \Longrightarrow X \lesssim Y$
⟨*proof*⟩

lemma *lepoll-trans* [*trans*]: $[\![X \lesssim Y;\ \ Y \lesssim Z]\!] \Longrightarrow X \lesssim Z$
⟨*proof*⟩

lemma *eq-lepoll-trans* [*trans*]: $[\![X \approx Y;\ \ Y \lesssim Z]\!] \Longrightarrow X \lesssim Z$
⟨*proof*⟩

lemma *lepoll-eq-trans* [*trans*]: $[\![X \lesssim Y;\ \ Y \approx Z]\!] \Longrightarrow X \lesssim Z$
⟨*proof*⟩

lemma *eqpollI*: $[\![X \lesssim Y;\ \ Y \lesssim X]\!] \Longrightarrow X \approx Y$
⟨*proof*⟩

lemma *eqpollE*:
$[\![X \approx Y;\ [\![X \lesssim Y;\ Y \lesssim X]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *eqpoll-iff*: $X \approx Y \longleftrightarrow X \lesssim Y \wedge Y \lesssim X$
⟨*proof*⟩

lemma *lepoll-0-is-0*: $A \lesssim 0 \Longrightarrow A = 0$
⟨*proof*⟩

lemmas *empty-lepollI* = *empty-subsetI* [*THEN subset-imp-lepoll*]

lemma *lepoll-0-iff*: $A \lesssim 0 \longleftrightarrow A=0$
⟨*proof*⟩

lemma *Un-lepoll-Un*:
$[\![A \lesssim B;\ C \lesssim D;\ B \cap D = 0]\!] \Longrightarrow A \cup C \lesssim B \cup D$
⟨*proof*⟩

lemmas *eqpoll-0-is-0* = *eqpoll-imp-lepoll* [*THEN lepoll-0-is-0*]

lemma *eqpoll-0-iff*: $A \approx 0 \longleftrightarrow A=0$
⟨*proof*⟩

lemma *eqpoll-disjoint-Un*:
$[\![A \approx B;\ \ C \approx D;\ \ A \cap C = 0;\ \ B \cap D = 0]\!]$
$\Longrightarrow A \cup C \approx B \cup D$
⟨*proof*⟩

23.2 lesspoll: contributions by Krzysztof Grabczewski

lemma *lesspoll-not-refl*: $\neg\ (i \prec i)$
⟨*proof*⟩

lemma *lesspoll-irrefl* [*elim*!]: $i \prec i \Longrightarrow P$
⟨*proof*⟩

lemma *lesspoll-imp-lepoll*: $A \prec B \Longrightarrow A \lesssim B$
⟨*proof*⟩

lemma *lepoll-well-ord*: $[\![A \lesssim B;\ \textit{well-ord}(B,r)]\!] \Longrightarrow \exists\, s.\ \textit{well-ord}(A,s)$
⟨*proof*⟩

lemma *lepoll-iff-leqpoll*: $A \lesssim B \longleftrightarrow A \prec B \mid A \approx B$
⟨*proof*⟩

lemma *inj-not-surj-succ*:
assumes *fi*: $f \in \textit{inj}(A,\ \textit{succ}(m))$ **and** *fns*: $f \notin \textit{surj}(A,\ \textit{succ}(m))$
shows $\exists f.\ f \in \textit{inj}(A,m)$
⟨*proof*⟩

lemma *lesspoll-trans* [*trans*]:
$[\![X \prec Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
⟨*proof*⟩

lemma *lesspoll-trans1* [*trans*]:
$[\![X \lesssim Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
⟨*proof*⟩

lemma *lesspoll-trans2* [*trans*]:
$[\![X \prec Y;\ Y \lesssim Z]\!] \Longrightarrow X \prec Z$
⟨*proof*⟩

lemma *eq-lesspoll-trans* [*trans*]:
$[\![X \approx Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
⟨*proof*⟩

lemma *lesspoll-eq-trans* [*trans*]:
$[\![X \prec Y;\ Y \approx Z]\!] \Longrightarrow X \prec Z$
⟨*proof*⟩

lemma *Least-equality*:
  $[\![P(i);\ \ Ord(i);\ \ \bigwedge x.\ x<i \Longrightarrow \neg P(x)]\!] \Longrightarrow (\mu\ x.\ P(x)) = i$
⟨*proof*⟩

lemma *LeastI*:
  assumes *P*: $P(i)$ **and** *i*: $Ord(i)$ **shows** $P(\mu\ x.\ P(x))$
⟨*proof*⟩

The proof is almost identical to the one above!

lemma *Least-le*:
  assumes *P*: $P(i)$ **and** *i*: $Ord(i)$ **shows** $(\mu\ x.\ P(x)) \le i$
⟨*proof*⟩

lemma *less-LeastE*: $[\![P(i);\ \ i < (\mu\ x.\ P(x))]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *LeastI2*:
  $[\![P(i);\ \ Ord(i);\ \ \bigwedge j.\ P(j) \Longrightarrow Q(j)]\!] \Longrightarrow Q(\mu\ j.\ P(j))$
⟨*proof*⟩

lemma *Least-0*:
  $[\![\neg\ (\exists\, i.\ Ord(i) \wedge P(i))]\!] \Longrightarrow (\mu\ x.\ P(x)) = 0$
⟨*proof*⟩

lemma *Ord-Least* [*intro*,*simp*,*TC*]: $Ord(\mu\ x.\ P(x))$
⟨*proof*⟩

23.3 Basic Properties of Cardinals

lemma *Least-cong*: $(\bigwedge y.\ P(y) \longleftrightarrow Q(y)) \Longrightarrow (\mu\ x.\ P(x)) = (\mu\ x.\ Q(x))$
⟨*proof*⟩

lemma *cardinal-cong*: $X \approx Y \Longrightarrow |X| = |Y|$
⟨*proof*⟩

lemma *well-ord-cardinal-eqpoll*:
  assumes *r*: *well-ord*(*A*,*r*) **shows** $|A| \approx A$
⟨*proof*⟩

lemmas *Ord-cardinal-eqpoll* = *well-ord-Memrel* [*THEN well-ord-cardinal-eqpoll*]

lemma *Ord-cardinal-idem*: $Ord(A) \Longrightarrow ||A|| = |A|$
⟨*proof*⟩

lemma *well-ord-cardinal-eqE*:
assumes *woX*: *well-ord*(X,r) **and** *woY*: *well-ord*(Y,s) **and** *eq*: $|X| = |Y|$
shows $X \approx Y$
⟨*proof*⟩

lemma *well-ord-cardinal-eqpoll-iff*:
$[\![$*well-ord*(X,r); *well-ord*(Y,s)$]\!] \Longrightarrow |X| = |Y| \longleftrightarrow X \approx Y$
⟨*proof*⟩

lemma *Ord-cardinal-le*: $Ord(i) \Longrightarrow |i| \leq i$
⟨*proof*⟩

lemma *Card-cardinal-eq*: $Card(K) \Longrightarrow |K| = K$
⟨*proof*⟩

lemma *CardI*: $[\![Ord(i);\ \bigwedge j.\ j<i \Longrightarrow \neg(j \approx i)]\!] \Longrightarrow Card(i)$
⟨*proof*⟩

lemma *Card-is-Ord*: $Card(i) \Longrightarrow Ord(i)$
⟨*proof*⟩

lemma *Card-cardinal-le*: $Card(K) \Longrightarrow K \leq |K|$
⟨*proof*⟩

lemma *Ord-cardinal* [*simp*,*intro*!]: $Ord(|A|)$
⟨*proof*⟩

The cardinals are the initial ordinals.

lemma *Card-iff-initial*: $Card(K) \longleftrightarrow Ord(K) \wedge (\forall j.\ j<K \longrightarrow \neg\ j \approx K)$
⟨*proof*⟩

lemma *lt-Card-imp-lesspoll*: $[\![Card(a);\ i<a]\!] \Longrightarrow i \prec a$
⟨*proof*⟩

lemma *Card-0*: $Card(0)$
⟨*proof*⟩

lemma *Card-Un*: $[\![Card(K);\ \ Card(L)]\!] \Longrightarrow Card(K \cup L)$
⟨*proof*⟩

lemma *Card-cardinal* [*iff*]: $Card(|A|)$
⟨*proof*⟩

lemma *cardinal-eq-lemma*:
assumes *i*:$|i| \le j$ **and** *j*: $j \le i$ **shows** $|j| = |i|$
⟨*proof*⟩

lemma *cardinal-mono*:
assumes *ij*: $i \le j$ **shows** $|i| \le |j|$
⟨*proof*⟩

Since we have $|succ(nat)| \le |nat|$, the converse of *cardinal-mono* fails!

lemma *cardinal-lt-imp-lt*: $[\![|i| < |j|;\ \ Ord(i);\ \ Ord(j)]\!] \Longrightarrow i < j$
⟨*proof*⟩

lemma *Card-lt-imp-lt*: $[\![|i| < K;\ \ Ord(i);\ \ Card(K)]\!] \Longrightarrow i < K$
⟨*proof*⟩

lemma *Card-lt-iff*: $[\![Ord(i);\ \ Card(K)]\!] \Longrightarrow (|i| < K) \longleftrightarrow (i < K)$
⟨*proof*⟩

lemma *Card-le-iff*: $[\![Ord(i);\ \ Card(K)]\!] \Longrightarrow (K \le |i|) \longleftrightarrow (K \le i)$
⟨*proof*⟩

lemma *well-ord-lepoll-imp-cardinal-le*:
assumes *wB*: *well-ord*(*B*,*r*) **and** *AB*: $A \lesssim B$
shows $|A| \le |B|$
⟨*proof*⟩

lemma *lepoll-cardinal-le*: $[\![A \lesssim i;\ Ord(i)]\!] \Longrightarrow |A| \le i$
⟨*proof*⟩

lemma *lepoll-Ord-imp-eqpoll*: $[\![A \lesssim i;\ Ord(i)]\!] \Longrightarrow |A| \approx A$
⟨*proof*⟩

lemma *lesspoll-imp-eqpoll*: $[\![A \prec i;\ Ord(i)]\!] \Longrightarrow |A| \approx A$
⟨*proof*⟩

lemma *cardinal-subset-Ord*: $[\![A<=i;\ Ord(i)]\!] \Longrightarrow |A| \subseteq i$
⟨*proof*⟩

23.4 The finite cardinals

lemma *cons-lepoll-consD*:
$[\![cons(u,A) \lesssim cons(v,B);\ \ u \notin A;\ \ v \notin B]\!] \Longrightarrow A \lesssim B$

⟨*proof*⟩

lemma *cons-eqpoll-consD*: ⟦*cons*(*u*,*A*) ≈ *cons*(*v*,*B*); *u*∉*A*; *v*∉*B*⟧ ⟹ *A* ≈ *B*
⟨*proof*⟩

lemma *succ-lepoll-succD*: *succ*(*m*) ≲ *succ*(*n*) ⟹ *m* ≲ *n*
⟨*proof*⟩

lemma *nat-lepoll-imp-le*:
m ∈ *nat* ⟹ *n* ∈ *nat* ⟹ *m* ≲ *n* ⟹ *m* ≤ *n*
⟨*proof*⟩

lemma *nat-eqpoll-iff*: ⟦*m* ∈ *nat*; *n* ∈ *nat*⟧ ⟹ *m* ≈ *n* ⟷ *m* = *n*
⟨*proof*⟩

lemma *nat-into-Card*:
assumes *n*: *n* ∈ *nat* **shows** *Card*(*n*)
⟨*proof*⟩

lemmas *cardinal-0* = *nat-0I* [*THEN nat-into-Card*, *THEN Card-cardinal-eq*, *iff*]
lemmas *cardinal-1* = *nat-1I* [*THEN nat-into-Card*, *THEN Card-cardinal-eq*, *iff*]

lemma *succ-lepoll-natE*: ⟦*succ*(*n*) ≲ *n*; *n* ∈ *nat*⟧ ⟹ *P*
⟨*proof*⟩

lemma *nat-lepoll-imp-ex-eqpoll-n*:
⟦*n* ∈ *nat*; *nat* ≲ *X*⟧ ⟹ ∃ *Y*. *Y* ⊆ *X* ∧ *n* ≈ *Y*
⟨*proof*⟩

lemma *lepoll-succ*: *i* ≲ *succ*(*i*)
⟨*proof*⟩

lemma *lepoll-imp-lesspoll-succ*:
assumes *A*: *A* ≲ *m* **and** *m*: *m* ∈ *nat*
shows *A* ≺ *succ*(*m*)
⟨*proof*⟩

lemma *lesspoll-succ-imp-lepoll*:
⟦*A* ≺ *succ*(*m*); *m* ∈ *nat*⟧ ⟹ *A* ≲ *m*
⟨*proof*⟩

lemma *lesspoll-succ-iff*: $m \in nat \Longrightarrow A \prec succ(m) \longleftrightarrow A \lesssim m$
⟨*proof*⟩

lemma *lepoll-succ-disj*: $[\![A \lesssim succ(m);\ \ m \in nat]\!] \Longrightarrow A \lesssim m \mid A \approx succ(m)$
⟨*proof*⟩

lemma *lesspoll-cardinal-lt*: $[\![A \prec i;\ Ord(i)]\!] \Longrightarrow |A| < i$
⟨*proof*⟩

23.5 The first infinite cardinal: Omega, or nat

lemma *lt-not-lepoll*:
assumes *n*: $n<i$ $n \in nat$ **shows** $\neg\ i \lesssim n$
⟨*proof*⟩

A slightly weaker version of *nat-eqpoll-iff*

lemma *Ord-nat-eqpoll-iff*:
assumes *i*: $Ord(i)$ **and** *n*: $n \in nat$ **shows** $i \approx n \longleftrightarrow i=n$
⟨*proof*⟩

lemma *Card-nat*: $Card(nat)$
⟨*proof*⟩

lemma *nat-le-cardinal*: $nat \le i \Longrightarrow nat \le |i|$
⟨*proof*⟩

lemma *n-lesspoll-nat*: $n \in nat \Longrightarrow n \prec nat$
⟨*proof*⟩

23.6 Towards Cardinal Arithmetic

lemma *cons-lepoll-cong*:
$[\![A \lesssim B;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \lesssim cons(b,B)$
⟨*proof*⟩

lemma *cons-eqpoll-cong*:
$[\![A \approx B;\ \ a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \approx cons(b,B)$
⟨*proof*⟩

lemma *cons-lepoll-cons-iff*:
$[\![a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \lesssim cons(b,B) \ \longleftrightarrow\ A \lesssim B$
⟨*proof*⟩

lemma *cons-eqpoll-cons-iff*:
$[\![a \notin A;\ \ b \notin B]\!] \Longrightarrow cons(a,A) \approx cons(b,B) \ \longleftrightarrow\ A \approx B$
⟨*proof*⟩

lemma *singleton-eqpoll-1*: $\{a\} \approx 1$

⟨*proof*⟩

lemma *cardinal-singleton*: $|\{a\}| = 1$
⟨*proof*⟩

lemma *not-0-is-lepoll-1*: $A \neq 0 \Longrightarrow 1 \lesssim A$
⟨*proof*⟩

lemma *succ-eqpoll-cong*: $A \approx B \Longrightarrow succ(A) \approx succ(B)$
⟨*proof*⟩

lemma *sum-eqpoll-cong*: $[\![A \approx C;\ B \approx D]\!] \Longrightarrow A+B \approx C+D$
⟨*proof*⟩

lemma *prod-eqpoll-cong*:
$[\![A \approx C;\ B \approx D]\!] \Longrightarrow A*B \approx C*D$
⟨*proof*⟩

lemma *inj-disjoint-eqpoll*:
$[\![f \in inj(A,B);\ A \cap B = 0]\!] \Longrightarrow A \cup (B - range(f)) \approx B$
⟨*proof*⟩

23.7 Lemmas by Krzysztof Grabczewski

If A has at most $n + 1$ elements and $a \in A$ then $A - \{a\}$ has at most n.

lemma *Diff-sing-lepoll*:
$[\![a \in A;\ A \lesssim succ(n)]\!] \Longrightarrow A - \{a\} \lesssim n$
⟨*proof*⟩

If A has at least $n + 1$ elements then $A - \{a\}$ has at least n.

lemma *lepoll-Diff-sing*:
assumes A: $succ(n) \lesssim A$ **shows** $n \lesssim A - \{a\}$
⟨*proof*⟩

lemma *Diff-sing-eqpoll*: $[\![a \in A;\ A \approx succ(n)]\!] \Longrightarrow A - \{a\} \approx n$
⟨*proof*⟩

lemma *lepoll-1-is-sing*: $[\![A \lesssim 1;\ a \in A]\!] \Longrightarrow A = \{a\}$
⟨*proof*⟩

lemma *Un-lepoll-sum*: $A \cup B \lesssim A+B$
⟨*proof*⟩

lemma *well-ord-Un*:
$[\![well\text{-}ord(X,R);\ well\text{-}ord(Y,S)]\!] \Longrightarrow \exists\, T.\ well\text{-}ord(X \cup Y,\ T)$
⟨*proof*⟩

lemma *disj-Un-eqpoll-sum*: $A \cap B = 0 \Longrightarrow A \cup B \approx A + B$
⟨*proof*⟩

23.8 Finite and infinite sets

lemma *eqpoll-imp-Finite-iff*: $A \approx B \Longrightarrow Finite(A) \longleftrightarrow Finite(B)$
⟨*proof*⟩

lemma *Finite-0* [*simp*]: $Finite(0)$
⟨*proof*⟩

lemma *Finite-cons*: $Finite(x) \Longrightarrow Finite(cons(y,x))$
⟨*proof*⟩

lemma *Finite-succ*: $Finite(x) \Longrightarrow Finite(succ(x))$
⟨*proof*⟩

lemma *lepoll-nat-imp-Finite*:
assumes A: $A \lesssim n$ **and** n: $n \in nat$ **shows** $Finite(A)$
⟨*proof*⟩

lemma *lesspoll-nat-is-Finite*:
$A \prec nat \Longrightarrow Finite(A)$
⟨*proof*⟩

lemma *lepoll-Finite*:
assumes Y: $Y \lesssim X$ **and** X: $Finite(X)$ **shows** $Finite(Y)$
⟨*proof*⟩

lemmas *subset-Finite* = *subset-imp-lepoll* [*THEN lepoll-Finite*]

lemma *Finite-cons-iff* [*iff*]: $Finite(cons(y,x)) \longleftrightarrow Finite(x)$
⟨*proof*⟩

lemma *Finite-succ-iff* [*iff*]: $Finite(succ(x)) \longleftrightarrow Finite(x)$
⟨*proof*⟩

lemma *Finite-Int*: $Finite(A) \mid Finite(B) \Longrightarrow Finite(A \cap B)$
⟨*proof*⟩

lemmas *Finite-Diff* = *Diff-subset* [*THEN subset-Finite*]

lemma *nat-le-infinite-Ord*:
$[\![Ord(i);\ \neg\ Finite(i)]\!] \Longrightarrow nat \le i$
⟨*proof*⟩

lemma *Finite-imp-well-ord*:

$Finite(A) \Longrightarrow \exists\, r.\ well\text{-}ord(A,r)$
⟨*proof*⟩

lemma *succ-lepoll-imp-not-empty*: $succ(x) \lesssim y \Longrightarrow y \neq 0$
⟨*proof*⟩

lemma *eqpoll-succ-imp-not-empty*: $x \approx succ(n) \Longrightarrow x \neq 0$
⟨*proof*⟩

lemma *Finite-Fin-lemma* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, A.\ (A \approx n \wedge A \subseteq X) \longrightarrow A \in Fin(X)$
⟨*proof*⟩

lemma *Finite-Fin*: $[\![Finite(A);\ A \subseteq X]\!] \Longrightarrow A \in Fin(X)$
⟨*proof*⟩

lemma *Fin-lemma* [*rule-format*]: $n \in nat \Longrightarrow \forall\, A.\ A \approx n \longrightarrow A \in Fin(A)$
⟨*proof*⟩

lemma *Finite-into-Fin*: $Finite(A) \Longrightarrow A \in Fin(A)$
⟨*proof*⟩

lemma *Fin-into-Finite*: $A \in Fin(U) \Longrightarrow Finite(A)$
⟨*proof*⟩

lemma *Finite-Fin-iff*: $Finite(A) \longleftrightarrow A \in Fin(A)$
⟨*proof*⟩

lemma *Finite-Un*: $[\![Finite(A);\ Finite(B)]\!] \Longrightarrow Finite(A \cup B)$
⟨*proof*⟩

lemma *Finite-Un-iff* [*simp*]: $Finite(A \cup B) \longleftrightarrow (Finite(A) \wedge Finite(B))$
⟨*proof*⟩

The converse must hold too.

lemma *Finite-Union*: $[\![\forall\, y{\in}X.\ Finite(y);\ \ Finite(X)]\!] \Longrightarrow Finite(\bigcup(X))$
⟨*proof*⟩

lemma *Finite-induct* [*case-names 0 cons*, *induct set*: *Finite*]:
$[\![Finite(A);\ P(0);$
$\quad \bigwedge x\ B.\ \ [\![Finite(B);\ x \notin B;\ P(B)]\!] \Longrightarrow P(cons(x,\ B))]\!]$
$\Longrightarrow P(A)$
⟨*proof*⟩

lemma *Diff-sing-Finite*: $Finite(A - \{a\}) \Longrightarrow Finite(A)$
⟨*proof*⟩

lemma *Diff-Finite* [*rule-format*]: $Finite(B) \Longrightarrow Finite(A-B) \longrightarrow Finite(A)$
⟨*proof*⟩

lemma *Finite-RepFun*: $Finite(A) \Longrightarrow Finite(RepFun(A,f))$
⟨*proof*⟩

lemma *Finite-RepFun-iff-lemma* [*rule-format*]:
$[\![Finite(x); \bigwedge x\ y.\ f(x)=f(y) \Longrightarrow x=y]\!]$
$\Longrightarrow \forall A.\ x = RepFun(A,f) \longrightarrow Finite(A)$
⟨*proof*⟩

I don't know why, but if the premise is expressed using meta-connectives then the simplifier cannot prove it automatically in conditional rewriting.

lemma *Finite-RepFun-iff*:
$(\forall x\ y.\ f(x)=f(y) \longrightarrow x=y) \Longrightarrow Finite(RepFun(A,f)) \longleftrightarrow Finite(A)$
⟨*proof*⟩

lemma *Finite-Pow*: $Finite(A) \Longrightarrow Finite(Pow(A))$
⟨*proof*⟩

lemma *Finite-Pow-imp-Finite*: $Finite(Pow(A)) \Longrightarrow Finite(A)$
⟨*proof*⟩

lemma *Finite-Pow-iff* [*iff*]: $Finite(Pow(A)) \longleftrightarrow Finite(A)$
⟨*proof*⟩

lemma *Finite-cardinal-iff*:
assumes *i*: $Ord(i)$ **shows** $Finite(|i|) \longleftrightarrow Finite(i)$
⟨*proof*⟩

lemma *nat-wf-on-converse-Memrel*: $n \in nat \Longrightarrow wf[n](converse(Memrel(n)))$
⟨*proof*⟩

lemma *nat-well-ord-converse-Memrel*: $n \in nat \Longrightarrow well\text{-}ord(n,converse(Memrel(n)))$
⟨*proof*⟩

lemma *well-ord-converse*:
$[\![well\text{-}ord(A,r);$
$well\text{-}ord(ordertype(A,r),\ converse(Memrel(ordertype(A,\ r))))]\!]$
$\Longrightarrow well\text{-}ord(A,converse(r))$
⟨*proof*⟩

lemma *ordertype-eq-n*:
assumes *r*: $well\text{-}ord(A,r)$ **and** *A*: $A \approx n$ **and** *n*: $n \in nat$
shows $ordertype(A,r) = n$

⟨*proof*⟩

lemma *Finite-well-ord-converse*:
  ⟦*Finite*(*A*); *well-ord*(*A*,*r*)⟧ ⟹ *well-ord*(*A*,*converse*(*r*))
⟨*proof*⟩

lemma *nat-into-Finite*: *n* ∈ *nat* ⟹ *Finite*(*n*)
⟨*proof*⟩

lemma *nat-not-Finite*: ¬ *Finite*(*nat*)
⟨*proof*⟩

end

24 The Cumulative Hierarchy and a Small Universe for Recursive Types

theory *Univ* **imports** *Epsilon Cardinal* **begin**

definition
  Vfrom :: [*i*,*i*]⇒*i* **where**
  Vfrom(*A*,*i*) ≡ *transrec*(*i*, λ*x f*. *A* ∪ (⋃*y*∈*x*. *Pow*(*f*‘*y*)))

abbreviation
  Vset :: *i*⇒*i* **where**
  Vset(*x*) ≡ *Vfrom*(*0*,*x*)

definition
  Vrec :: [*i*, [*i*,*i*]⇒*i*] ⇒*i* **where**
  Vrec(*a*,*H*) ≡ *transrec*(*rank*(*a*), λ*x g*. λ*z*∈*Vset*(*succ*(*x*)).
                    H(*z*, λ*w*∈*Vset*(*x*). *g*‘*rank*(*w*)‘*w*)) ‘ *a*

definition
  Vrecursor :: [[*i*,*i*]⇒*i*, *i*] ⇒*i* **where**
  Vrecursor(*H*,*a*) ≡ *transrec*(*rank*(*a*), λ*x g*. λ*z*∈*Vset*(*succ*(*x*)).
                        H(λ*w*∈*Vset*(*x*). *g*‘*rank*(*w*)‘*w*, *z*)) ‘ *a*

definition
  univ :: *i*⇒*i* **where**
  univ(*A*) ≡ *Vfrom*(*A*,*nat*)

24.1 Immediate Consequences of the Definition of *Vfrom*(*A*, *i*)

NOT SUITABLE FOR REWRITING – RECURSIVE!

lemma *Vfrom*: *Vfrom*(*A*,*i*) = *A* ∪ (⋃*j*∈*i*. *Pow*(*Vfrom*(*A*,*j*)))
⟨*proof*⟩

24.1.1 Monotonicity

lemma *Vfrom-mono* [*rule-format*]:
$A<=B \Longrightarrow \forall j.\ i<=j \longrightarrow Vfrom(A,i) \subseteq Vfrom(B,j)$
⟨*proof*⟩

lemma *VfromI*: $[\![a \in Vfrom(A,j);\ j<i]\!] \Longrightarrow a \in Vfrom(A,i)$
⟨*proof*⟩

24.1.2 A fundamental equality: Vfrom does not require ordinals!

lemma *Vfrom-rank-subset1*: $Vfrom(A,x) \subseteq Vfrom(A,rank(x))$
⟨*proof*⟩

lemma *Vfrom-rank-subset2*: $Vfrom(A,rank(x)) \subseteq Vfrom(A,x)$
⟨*proof*⟩

lemma *Vfrom-rank-eq*: $Vfrom(A,rank(x)) = Vfrom(A,x)$
⟨*proof*⟩

24.2 Basic Closure Properties

lemma *zero-in-Vfrom*: $y{:}x \Longrightarrow 0 \in Vfrom(A,x)$
⟨*proof*⟩

lemma *i-subset-Vfrom*: $i \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemma *A-subset-Vfrom*: $A \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemmas *A-into-Vfrom* = *A-subset-Vfrom* [*THEN subsetD*]

lemma *subset-mem-Vfrom*: $a \subseteq Vfrom(A,i) \Longrightarrow a \in Vfrom(A,succ(i))$
⟨*proof*⟩

24.2.1 Finite sets and ordered pairs

lemma *singleton-in-Vfrom*: $a \in Vfrom(A,i) \Longrightarrow \{a\} \in Vfrom(A,succ(i))$
⟨*proof*⟩

lemma *doubleton-in-Vfrom*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i)]\!] \Longrightarrow \{a,b\} \in Vfrom(A,succ(i))$
⟨*proof*⟩

lemma *Pair-in-Vfrom*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i)]\!] \Longrightarrow \langle a,b\rangle \in Vfrom(A,succ(succ(i)))$
⟨*proof*⟩

lemma *succ-in-Vfrom*: $a \subseteq Vfrom(A,i) \Longrightarrow succ(a) \in Vfrom(A,succ(succ(i)))$

⟨*proof*⟩

24.3 0, Successor and Limit Equations for *Vfrom*

lemma *Vfrom-0*: $Vfrom(A,0) = A$
⟨*proof*⟩

lemma *Vfrom-succ-lemma*: $Ord(i) \Longrightarrow Vfrom(A,succ(i)) = A \cup Pow(Vfrom(A,i))$
⟨*proof*⟩

lemma *Vfrom-succ*: $Vfrom(A,succ(i)) = A \cup Pow(Vfrom(A,i))$
⟨*proof*⟩

lemma *Vfrom-Union*: $y{:}X \Longrightarrow Vfrom(A,\bigcup(X)) = (\bigcup y{\in}X.\ Vfrom(A,y))$
⟨*proof*⟩

24.4 *Vfrom* applied to Limit Ordinals

lemma *Limit-Vfrom-eq*:
$Limit(i) \Longrightarrow Vfrom(A,i) = (\bigcup y{\in}i.\ Vfrom(A,y))$
⟨*proof*⟩

lemma *Limit-VfromE*:
$[\![a \in Vfrom(A,i);\ \neg R \Longrightarrow Limit(i);$
$\bigwedge x.\ [\![x{<}i;\ a \in Vfrom(A,x)]\!] \Longrightarrow R$
$]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *singleton-in-VLimit*:
$[\![a \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow \{a\} \in Vfrom(A,i)$
⟨*proof*⟩

lemmas *Vfrom-UnI1* =
Un-upper1 [*THEN subset-refl* [*THEN Vfrom-mono*, *THEN subsetD*]]
lemmas *Vfrom-UnI2* =
Un-upper2 [*THEN subset-refl* [*THEN Vfrom-mono*, *THEN subsetD*]]

Hard work is finding a single j:i such that a,b<=Vfrom(A,j)

lemma *doubleton-in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow \{a,b\} \in Vfrom(A,i)$
⟨*proof*⟩

lemma *Pair-in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow \langle a,b\rangle \in Vfrom(A,i)$

Infer that a, b occur at ordinals x,xa < i.

⟨*proof*⟩

lemma *product-VLimit*: $Limit(i) \Longrightarrow Vfrom(A,i) * Vfrom(A,i) \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemmas *Sigma-subset-VLimit* =
subset-trans [*OF Sigma-mono product-VLimit*]

lemmas *nat-subset-VLimit* =
subset-trans [*OF nat-le-Limit* [*THEN le-imp-subset*] *i-subset-Vfrom*]

lemma *nat-into-VLimit*: $[\![n: nat;\ Limit(i)]\!] \Longrightarrow n \in Vfrom(A,i)$
⟨*proof*⟩

24.4.1 Closure under Disjoint Union

lemmas *zero-in-VLimit* = *Limit-has-0* [*THEN ltD*, *THEN zero-in-Vfrom*]

lemma *one-in-VLimit*: $Limit(i) \Longrightarrow 1 \in Vfrom(A,i)$
⟨*proof*⟩

lemma *Inl-in-VLimit*:
$[\![a \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow Inl(a) \in Vfrom(A,i)$
⟨*proof*⟩

lemma *Inr-in-VLimit*:
$[\![b \in Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow Inr(b) \in Vfrom(A,i)$
⟨*proof*⟩

lemma *sum-VLimit*: $Limit(i) \Longrightarrow Vfrom(C,i)+Vfrom(C,i) \subseteq Vfrom(C,i)$
⟨*proof*⟩

lemmas *sum-subset-VLimit* = *subset-trans* [*OF sum-mono sum-VLimit*]

24.5 Properties assuming $Transset(A)$

lemma *Transset-Vfrom*: $Transset(A) \Longrightarrow Transset(Vfrom(A,i))$
⟨*proof*⟩

lemma *Transset-Vfrom-succ*:
$Transset(A) \Longrightarrow Vfrom(A,\ succ(i)) = Pow(Vfrom(A,i))$
⟨*proof*⟩

lemma *Transset-Pair-subset*: $[\![\langle a,b\rangle \subseteq C;\ Transset(C)]\!] \Longrightarrow a: C \wedge b: C$
⟨*proof*⟩

lemma *Transset-Pair-subset-VLimit*:
$[\![\langle a,b\rangle \subseteq Vfrom(A,i);\ Transset(A);\ Limit(i)]\!]$
$\Longrightarrow \langle a,b\rangle \in Vfrom(A,i)$
⟨*proof*⟩

lemma *Union-in-Vfrom*:

$[\![X \in Vfrom(A,j);\ Transset(A)]\!] \Longrightarrow \bigcup(X) \in Vfrom(A,\ succ(j))$
⟨*proof*⟩

lemma *Union-in-VLimit*:
$[\![X \in Vfrom(A,i);\ Limit(i);\ Transset(A)]\!] \Longrightarrow \bigcup(X) \in Vfrom(A,i)$
⟨*proof*⟩

General theorem for membership in Vfrom(A,i) when i is a limit ordinal

lemma *in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i);$
$\bigwedge x\ y\ j.\ [\![j<i;\ 1{:}j;\ x \in Vfrom(A,j);\ y \in Vfrom(A,j)]\!]$
$\Longrightarrow \exists k.\ h(x,y) \in Vfrom(A,k) \wedge k<i]\!]$
$\Longrightarrow h(a,b) \in Vfrom(A,i)$

Infer that a, b occur at ordinals x,xa < i.

⟨*proof*⟩

24.5.1 Products

lemma *prod-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ b \in Vfrom(A,j);\ Transset(A)]\!]$
$\Longrightarrow a*b \in Vfrom(A,\ succ(succ(succ(j))))$
⟨*proof*⟩

lemma *prod-in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i);\ Transset(A)]\!]$
$\Longrightarrow a*b \in Vfrom(A,i)$
⟨*proof*⟩

24.5.2 Disjoint Sums, or Quine Ordered Pairs

lemma *sum-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ b \in Vfrom(A,j);\ Transset(A);\ 1{:}j]\!]$
$\Longrightarrow a+b \in Vfrom(A,\ succ(succ(succ(j))))$
⟨*proof*⟩

lemma *sum-in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i);\ Transset(A)]\!]$
$\Longrightarrow a+b \in Vfrom(A,i)$
⟨*proof*⟩

24.5.3 Function Space!

lemma *fun-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ b \in Vfrom(A,j);\ Transset(A)]\!] \Longrightarrow$
$a->b \in Vfrom(A,\ succ(succ(succ(succ(j)))))$
⟨*proof*⟩

lemma *fun-in-VLimit*:
$[\![a \in Vfrom(A,i);\ b \in Vfrom(A,i);\ Limit(i);\ Transset(A)]\!]$

$\Longrightarrow a->b \in Vfrom(A,i)$
⟨*proof*⟩

lemma *Pow-in-Vfrom*:
$[\![a \in Vfrom(A,j);\ \ Transset(A)]\!] \Longrightarrow Pow(a) \in Vfrom(A,\ succ(succ(j)))$
⟨*proof*⟩

lemma *Pow-in-VLimit*:
$[\![a \in Vfrom(A,i);\ \ Limit(i);\ \ Transset(A)]\!] \Longrightarrow Pow(a) \in Vfrom(A,i)$
⟨*proof*⟩

24.6 The Set $Vset(i)$

lemma *Vset*: $Vset(i) = (\bigcup j{\in}i.\ Pow(Vset(j)))$
⟨*proof*⟩

lemmas *Vset-succ* = *Transset-0* [*THEN Transset-Vfrom-succ*]
lemmas *Transset-Vset* = *Transset-0* [*THEN Transset-Vfrom*]

24.6.1 Characterisation of the elements of $Vset(i)$

lemma *VsetD* [*rule-format*]: $Ord(i) \Longrightarrow \forall\, b.\ b \in Vset(i) \longrightarrow rank(b) < i$
⟨*proof*⟩

lemma *VsetI-lemma* [*rule-format*]:
$Ord(i) \Longrightarrow \forall\, b.\ rank(b) \in i \longrightarrow b \in Vset(i)$
⟨*proof*⟩

lemma *VsetI*: $rank(x){<}i \Longrightarrow x \in Vset(i)$
⟨*proof*⟩

Merely a lemma for the next result

lemma *Vset-Ord-rank-iff*: $Ord(i) \Longrightarrow b \in Vset(i) \longleftrightarrow rank(b) < i$
⟨*proof*⟩

lemma *Vset-rank-iff* [*simp*]: $b \in Vset(a) \longleftrightarrow rank(b) < rank(a)$
⟨*proof*⟩

This is rank(rank(a)) = rank(a)

declare *Ord-rank* [*THEN rank-of-Ord*, *simp*]

lemma *rank-Vset*: $Ord(i) \Longrightarrow rank(Vset(i)) = i$
⟨*proof*⟩

lemma *Finite-Vset*: $i \in nat \Longrightarrow Finite(Vset(i))$
⟨*proof*⟩

24.6.2 Reasoning about Sets in Terms of Their Elements' Ranks

lemma *arg-subset-Vset-rank*: $a \subseteq Vset(rank(a))$

⟨*proof*⟩

lemma *Int-Vset-subset*:
$[\![\bigwedge i.\ Ord(i) \Longrightarrow a \cap Vset(i) \subseteq b]\!] \Longrightarrow a \subseteq b$
⟨*proof*⟩

24.6.3 Set Up an Environment for Simplification

lemma *rank-Inl*: $rank(a) < rank(Inl(a))$
⟨*proof*⟩

lemma *rank-Inr*: $rank(a) < rank(Inr(a))$
⟨*proof*⟩

lemmas *rank-rls* = *rank-Inl rank-Inr rank-pair1 rank-pair2*

24.6.4 Recursion over Vset Levels!

NOT SUITABLE FOR REWRITING: recursive!

lemma *Vrec*: $Vrec(a,H) = H(a,\ \lambda x{\in}Vset(rank(a)).\ Vrec(x,H))$
⟨*proof*⟩

This form avoids giant explosions in proofs. NOTE the form of the premise!

lemma *def-Vrec*:
$[\![\bigwedge x.\ h(x){\equiv}Vrec(x,H)]\!] \Longrightarrow$
$h(a) = H(a,\ \lambda x{\in}Vset(rank(a)).\ h(x))$
⟨*proof*⟩

NOT SUITABLE FOR REWRITING: recursive!

lemma *Vrecursor*:
$Vrecursor(H,a) = H(\lambda x{\in}Vset(rank(a)).\ Vrecursor(H,x),\ \ a)$
⟨*proof*⟩

This form avoids giant explosions in proofs. NOTE the form of the premise!

lemma *def-Vrecursor*:
$h \equiv Vrecursor(H) \Longrightarrow h(a) = H(\lambda x{\in}Vset(rank(a)).\ h(x),\ \ a)$
⟨*proof*⟩

24.7 The Datatype Universe: *univ*(*A*)

lemma *univ-mono*: $A{<=}B \Longrightarrow univ(A) \subseteq univ(B)$
⟨*proof*⟩

lemma *Transset-univ*: $Transset(A) \Longrightarrow Transset(univ(A))$
⟨*proof*⟩

24.7.1 The Set $univ(A)$ as a Limit

lemma *univ-eq-UN*: $univ(A) = (\bigcup i{\in}nat.\ Vfrom(A,i))$
⟨*proof*⟩

lemma *subset-univ-eq-Int*: $c \subseteq univ(A) \Longrightarrow c = (\bigcup i{\in}nat.\ c \cap Vfrom(A,i))$
⟨*proof*⟩

lemma *univ-Int-Vfrom-subset*:
$[\![a \subseteq univ(X);$
$\bigwedge i.\ i{:}nat \Longrightarrow a \cap Vfrom(X,i) \subseteq b]\!]$
$\Longrightarrow a \subseteq b$
⟨*proof*⟩

lemma *univ-Int-Vfrom-eq*:
$[\![a \subseteq univ(X);\quad b \subseteq univ(X);$
$\bigwedge i.\ i{:}nat \Longrightarrow a \cap Vfrom(X,i) = b \cap Vfrom(X,i)$
$]\!] \Longrightarrow a = b$
⟨*proof*⟩

24.8 Closure Properties for $univ(A)$

lemma *zero-in-univ*: $0 \in univ(A)$
⟨*proof*⟩

lemma *zero-subset-univ*: $\{0\} \subseteq univ(A)$
⟨*proof*⟩

lemma *A-subset-univ*: $A \subseteq univ(A)$
⟨*proof*⟩

lemmas *A-into-univ* = *A-subset-univ* [*THEN subsetD*]

24.8.1 Closure under Unordered and Ordered Pairs

lemma *singleton-in-univ*: $a{:}\ univ(A) \Longrightarrow \{a\} \in univ(A)$
⟨*proof*⟩

lemma *doubleton-in-univ*:
$[\![a{:}\ univ(A);\ \ b{:}\ univ(A)]\!] \Longrightarrow \{a,b\} \in univ(A)$
⟨*proof*⟩

lemma *Pair-in-univ*:
$[\![a{:}\ univ(A);\ \ b{:}\ univ(A)]\!] \Longrightarrow \langle a,b\rangle \in univ(A)$
⟨*proof*⟩

lemma *Union-in-univ*:
$[\![X{:}\ univ(A);\ \ Transset(A)]\!] \Longrightarrow \bigcup(X) \in univ(A)$
⟨*proof*⟩

lemma *product-univ*: $univ(A)*univ(A) \subseteq univ(A)$
⟨*proof*⟩

24.8.2 The Natural Numbers

lemma *nat-subset-univ*: $nat \subseteq univ(A)$
⟨*proof*⟩

lemma *nat-into-univ*: $n \in nat \Longrightarrow n \in univ(A)$
⟨*proof*⟩

24.8.3 Instances for 1 and 2

lemma *one-in-univ*: $1 \in univ(A)$
⟨*proof*⟩

unused!

lemma *two-in-univ*: $2 \in univ(A)$
⟨*proof*⟩

lemma *bool-subset-univ*: $bool \subseteq univ(A)$
⟨*proof*⟩

lemmas *bool-into-univ* = *bool-subset-univ* [*THEN subsetD*]

24.8.4 Closure under Disjoint Union

lemma *Inl-in-univ*: a: $univ(A) \Longrightarrow Inl(a) \in univ(A)$
⟨*proof*⟩

lemma *Inr-in-univ*: b: $univ(A) \Longrightarrow Inr(b) \in univ(A)$
⟨*proof*⟩

lemma *sum-univ*: $univ(C)+univ(C) \subseteq univ(C)$
⟨*proof*⟩

lemmas *sum-subset-univ* = *subset-trans* [*OF sum-mono sum-univ*]

lemma *Sigma-subset-univ*:
$[\![A \subseteq univ(D); \bigwedge x.\ x \in A \Longrightarrow B(x) \subseteq univ(D)]\!] \Longrightarrow Sigma(A,B) \subseteq univ(D)$
⟨*proof*⟩

24.9 Finite Branching Closure Properties

24.9.1 Closure under Finite Powerset

lemma *Fin-Vfrom-lemma*:
$[\![b$: $Fin(Vfrom(A,i));\ \ Limit(i)]\!] \Longrightarrow \exists j.\ b \subseteq Vfrom(A,j) \wedge j<i$
⟨*proof*⟩

lemma *Fin-VLimit*: $Limit(i) \Longrightarrow Fin(Vfrom(A,i)) \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemmas *Fin-subset-VLimit* = *subset-trans* [*OF Fin-mono Fin-VLimit*]

lemma *Fin-univ*: $Fin(univ(A)) \subseteq univ(A)$
⟨*proof*⟩

24.9.2 Closure under Finite Powers: Functions from a Natural Number

lemma *nat-fun-VLimit*:
$[\![n: nat;\ Limit(i)]\!] \Longrightarrow n -> Vfrom(A,i) \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemmas *nat-fun-subset-VLimit* = *subset-trans* [*OF Pi-mono nat-fun-VLimit*]

lemma *nat-fun-univ*: $n: nat \Longrightarrow n -> univ(A) \subseteq univ(A)$
⟨*proof*⟩

24.9.3 Closure under Finite Function Space

General but seldom-used version; normally the domain is fixed

lemma *FiniteFun-VLimit1*:
$Limit(i) \Longrightarrow Vfrom(A,i) -||> Vfrom(A,i) \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemma *FiniteFun-univ1*: $univ(A) -||> univ(A) \subseteq univ(A)$
⟨*proof*⟩

Version for a fixed domain

lemma *FiniteFun-VLimit*:
$[\![W \subseteq Vfrom(A,i);\ Limit(i)]\!] \Longrightarrow W -||> Vfrom(A,i) \subseteq Vfrom(A,i)$
⟨*proof*⟩

lemma *FiniteFun-univ*:
$W \subseteq univ(A) \Longrightarrow W -||> univ(A) \subseteq univ(A)$
⟨*proof*⟩

lemma *FiniteFun-in-univ*:
$[\![f: W -||> univ(A);\ W \subseteq univ(A)]\!] \Longrightarrow f \in univ(A)$
⟨*proof*⟩

Remove $\subseteq$ from the rule above

lemmas *FiniteFun-in-univ'* = *FiniteFun-in-univ* [*OF - subsetI*]

24.10 * For QUniv. Properties of Vfrom analogous to the "take-lemma" *

Intersecting a*b with Vfrom...

This version says a, b exist one level down, in the smaller set Vfrom(X,i)

lemma *doubleton-in-Vfrom-D*:
$[\![\{a,b\} \in Vfrom(X,succ(i));\ \ Transset(X)]\!]$
$\Longrightarrow a \in Vfrom(X,i) \ \wedge \ b \in Vfrom(X,i)$
⟨*proof*⟩

This weaker version says a, b exist at the same level

lemmas *Vfrom-doubleton-D* = *Transset-Vfrom* [*THEN Transset-doubleton-D*]

lemma *Pair-in-Vfrom-D*:
$[\![\langle a,b\rangle \in Vfrom(X,succ(i));\ \ Transset(X)]\!]$
$\Longrightarrow a \in Vfrom(X,i) \ \wedge \ b \in Vfrom(X,i)$
⟨*proof*⟩

lemma *product-Int-Vfrom-subset*:
$Transset(X) \Longrightarrow$
$(a*b) \cap Vfrom(X,\ succ(i)) \subseteq (a \cap Vfrom(X,i)) * (b \cap Vfrom(X,i))$
⟨*proof*⟩

⟨*ML*⟩

end

25 A Small Universe for Lazy Recursive Types

theory *QUniv* **imports** *Univ QPair* **begin**

rep-datatype
elimination *sumE*
induction *TrueI*
case-eqns *case-Inl case-Inr*

rep-datatype
elimination *qsumE*
induction *TrueI*
case-eqns *qcase-QInl qcase-QInr*

definition

quniv :: $i \Rightarrow i$ **where**
$quniv(A) \equiv Pow(univ(eclose(A)))$

25.1 Properties involving Transset and Sum

lemma *Transset-includes-summands*:
$[\![Transset(C);\ A+B \subseteq C]\!] \Longrightarrow A \subseteq C \wedge B \subseteq C$
⟨*proof*⟩

lemma *Transset-sum-Int-subset*:
$Transset(C) \Longrightarrow (A+B) \cap C \subseteq (A \cap C) + (B \cap C)$
⟨*proof*⟩

25.2 Introduction and Elimination Rules

lemma *qunivI*: $X \subseteq univ(eclose(A)) \Longrightarrow X \in quniv(A)$
⟨*proof*⟩

lemma *qunivD*: $X \in quniv(A) \Longrightarrow X \subseteq univ(eclose(A))$
⟨*proof*⟩

lemma *quniv-mono*: $A<=B \Longrightarrow quniv(A) \subseteq quniv(B)$
⟨*proof*⟩

25.3 Closure Properties

lemma *univ-eclose-subset-quniv*: $univ(eclose(A)) \subseteq quniv(A)$
⟨*proof*⟩

lemma *univ-subset-quniv*: $univ(A) \subseteq quniv(A)$
⟨*proof*⟩

lemmas *univ-into-quniv* = *univ-subset-quniv* [*THEN subsetD*]

lemma *Pow-univ-subset-quniv*: $Pow(univ(A)) \subseteq quniv(A)$
⟨*proof*⟩

lemmas *univ-subset-into-quniv* =
PowI [*THEN Pow-univ-subset-quniv* [*THEN subsetD*]]

lemmas *zero-in-quniv* = *zero-in-univ* [*THEN univ-into-quniv*]
lemmas *one-in-quniv* = *one-in-univ* [*THEN univ-into-quniv*]
lemmas *two-in-quniv* = *two-in-univ* [*THEN univ-into-quniv*]

lemmas *A-subset-quniv* = *subset-trans* [*OF A-subset-univ univ-subset-quniv*]

lemmas *A-into-quniv* = *A-subset-quniv* [*THEN subsetD*]

lemma *QPair-subset-univ*:
$[\![a \subseteq univ(A);\ \ b \subseteq univ(A)]\!] \Longrightarrow <a;b> \subseteq univ(A)$
⟨*proof*⟩

25.4 Quine Disjoint Sum

lemma *QInl-subset-univ*: $a \subseteq univ(A) \Longrightarrow QInl(a) \subseteq univ(A)$
⟨*proof*⟩

lemmas *naturals-subset-nat* =
Ord-nat [*THEN Ord-is-Transset*, *unfolded Transset-def*, *THEN bspec*]

lemmas *naturals-subset-univ* =
subset-trans [*OF naturals-subset-nat nat-subset-univ*]

lemma *QInr-subset-univ*: $a \subseteq univ(A) \Longrightarrow QInr(a) \subseteq univ(A)$
⟨*proof*⟩

25.5 Closure for Quine-Inspired Products and Sums

lemma *QPair-in-quniv*:
$[\![a: quniv(A);\ \ b: quniv(A)]\!] \Longrightarrow <a;b> \in quniv(A)$
⟨*proof*⟩

lemma *QSigma-quniv*: $quniv(A) <*> quniv(A) \subseteq quniv(A)$
⟨*proof*⟩

lemmas *QSigma-subset-quniv* = *subset-trans* [*OF QSigma-mono QSigma-quniv*]

lemma *quniv-QPair-D*:
$<a;b> \in quniv(A) \Longrightarrow a: quniv(A) \wedge b: quniv(A)$
⟨*proof*⟩

lemmas *quniv-QPair-E* = *quniv-QPair-D* [*THEN conjE*]

lemma *quniv-QPair-iff*: $<a;b> \in quniv(A) \longleftrightarrow a: quniv(A) \wedge b: quniv(A)$
⟨*proof*⟩

25.6 Quine Disjoint Sum

lemma *QInl-in-quniv*: $a: quniv(A) \Longrightarrow QInl(a) \in quniv(A)$
⟨*proof*⟩

lemma *QInr-in-quniv*: $b: quniv(A) \Longrightarrow QInr(b) \in quniv(A)$
⟨*proof*⟩

lemma *qsum-quniv*: $quniv(C) <+> quniv(C) \subseteq quniv(C)$

⟨*proof*⟩

lemmas *qsum-subset-quniv* = *subset-trans* [*OF qsum-mono qsum-quniv*]

25.7 The Natural Numbers

lemmas *nat-subset-quniv* = *subset-trans* [*OF nat-subset-univ univ-subset-quniv*]

lemmas *nat-into-quniv* = *nat-subset-quniv* [*THEN subsetD*]

lemmas *bool-subset-quniv* = *subset-trans* [*OF bool-subset-univ univ-subset-quniv*]

lemmas *bool-into-quniv* = *bool-subset-quniv* [*THEN subsetD*]

lemma *QPair-Int-Vfrom-succ-subset*:
Transset(*X*) ⟹
<*a*;*b*> ∩ *Vfrom*(*X*, *succ*(*i*)) ⊆ <*a* ∩ *Vfrom*(*X*,*i*); *b* ∩ *Vfrom*(*X*,*i*)>
⟨*proof*⟩

25.8 "Take-Lemma" Rules

lemma *QPair-Int-Vfrom-subset*:
Transset(*X*) ⟹
<*a*;*b*> ∩ *Vfrom*(*X*,*i*) ⊆ <*a* ∩ *Vfrom*(*X*,*i*); *b* ∩ *Vfrom*(*X*,*i*)>
⟨*proof*⟩

lemmas *QPair-Int-Vset-subset-trans* =
subset-trans [*OF Transset-0* [*THEN QPair-Int-Vfrom-subset*] *QPair-mono*]

lemma *QPair-Int-Vset-subset-UN*:
Ord(*i*) ⟹ <*a*;*b*> ∩ *Vset*(*i*) ⊆ (⋃*j*∈*i*. <*a* ∩ *Vset*(*j*); *b* ∩ *Vset*(*j*)>)
⟨*proof*⟩

end

26 Datatype and CoDatatype Definitions

theory *Datatype*
imports *Inductive Univ QUniv*
keywords *datatype codatatype* :: *thy-decl*
begin

⟨*ML*⟩

end

27 Arithmetic Operators and Their Definitions

theory *Arith* **imports** *Univ* **begin**

Proofs about elementary arithmetic: addition, multiplication, etc.

definition
  pred :: $i \Rightarrow i$ **where**
    $pred(y) \equiv nat\text{-}case(0,\ \lambda x.\ x,\ y)$

definition
  natify :: $i \Rightarrow i$ **where**
    $natify \equiv Vrecursor(\lambda f\ a.\ if\ a = succ(pred(a))\ then\ succ(f`pred(a))\ else\ 0)$

consts
  raw-add :: $[i,i] \Rightarrow i$
  raw-diff :: $[i,i] \Rightarrow i$
  raw-mult :: $[i,i] \Rightarrow i$

primrec
  $raw\text{-}add\ (0,\ n) = n$
  $raw\text{-}add\ (succ(m),\ n) = succ(raw\text{-}add(m,\ n))$

primrec
  raw-diff-0: $raw\text{-}diff(m,\ 0) = m$
  raw-diff-succ: $raw\text{-}diff(m,\ succ(n)) = nat\text{-}case(0,\ \lambda x.\ x,\ raw\text{-}diff(m,\ n))$

primrec
  $raw\text{-}mult(0,\ n) = 0$
  $raw\text{-}mult(succ(m),\ n) = raw\text{-}add\ (n,\ raw\text{-}mult(m,\ n))$

definition
  add :: $[i,i] \Rightarrow i$ (**infixl** ‹#+› 65) **where**
    $m\ \#+\ n \equiv raw\text{-}add\ (natify(m),\ natify(n))$

definition
  diff :: $[i,i] \Rightarrow i$ (**infixl** ‹#−› 65) **where**
    $m\ \#-\ n \equiv raw\text{-}diff\ (natify(m),\ natify(n))$

definition
  mult :: $[i,i] \Rightarrow i$ (**infixl** ‹#*› 70) **where**
    $m\ \#*\ n \equiv raw\text{-}mult\ (natify(m),\ natify(n))$

definition
  raw-div :: $[i,i] \Rightarrow i$ **where**
    $raw\text{-}div\ (m,\ n) \equiv$

transrec(*m*, λ*j f*. *if j*<*n* | *n*=*0 then 0 else succ*(*f*‘(*j*#−*n*)))

definition
raw-mod :: [*i*,*i*]⇒*i* **where**
raw-mod (*m*, *n*) ≡
transrec(*m*, λ*j f*. *if j*<*n* | *n*=*0 then j else f*‘(*j*#−*n*))

definition
div :: [*i*,*i*]⇒*i* (**infixl** ‹*div*› *70*) **where**
m div n ≡ *raw-div* (*natify*(*m*), *natify*(*n*))

definition
mod :: [*i*,*i*]⇒*i* (**infixl** ‹*mod*› *70*) **where**
m mod n ≡ *raw-mod* (*natify*(*m*), *natify*(*n*))

declare *rec-type* [*simp*]
nat-0-le [*simp*]

lemma *zero-lt-lemma*: ⟦*0*<*k*; *k* ∈ *nat*⟧ ⟹ ∃ *j*∈*nat*. *k* = *succ*(*j*)
⟨*proof*⟩

lemmas *zero-lt-natE* = *zero-lt-lemma* [*THEN bexE*]

27.1 *natify*, the Coercion to *nat*

lemma *pred-succ-eq* [*simp*]: *pred*(*succ*(*y*)) = *y*
⟨*proof*⟩

lemma *natify-succ*: *natify*(*succ*(*x*)) = *succ*(*natify*(*x*))
⟨*proof*⟩

lemma *natify-0* [*simp*]: *natify*(*0*) = *0*
⟨*proof*⟩

lemma *natify-non-succ*: ∀ *z*. *x* ≠ *succ*(*z*) ⟹ *natify*(*x*) = *0*
⟨*proof*⟩

lemma *natify-in-nat* [*iff*,*TC*]: *natify*(*x*) ∈ *nat*
⟨*proof*⟩

lemma *natify-ident* [*simp*]: *n* ∈ *nat* ⟹ *natify*(*n*) = *n*
⟨*proof*⟩

lemma *natify-eqE*: ⟦*natify*(*x*) = *y*; *x* ∈ *nat*⟧ ⟹ *x*=*y*
⟨*proof*⟩

lemma *natify-idem* [*simp*]: $natify(natify(x)) = natify(x)$
⟨*proof*⟩

lemma *add-natify1* [*simp*]: $natify(m) \#+ n = m \#+ n$
⟨*proof*⟩

lemma *add-natify2* [*simp*]: $m \#+ natify(n) = m \#+ n$
⟨*proof*⟩

lemma *mult-natify1* [*simp*]: $natify(m) \#* n = m \#* n$
⟨*proof*⟩

lemma *mult-natify2* [*simp*]: $m \#* natify(n) = m \#* n$
⟨*proof*⟩

lemma *diff-natify1* [*simp*]: $natify(m) \#- n = m \#- n$
⟨*proof*⟩

lemma *diff-natify2* [*simp*]: $m \#- natify(n) = m \#- n$
⟨*proof*⟩

lemma *mod-natify1* [*simp*]: $natify(m)\ mod\ n = m\ mod\ n$
⟨*proof*⟩

lemma *mod-natify2* [*simp*]: $m\ mod\ natify(n) = m\ mod\ n$
⟨*proof*⟩

lemma *div-natify1* [*simp*]: $natify(m)\ div\ n = m\ div\ n$
⟨*proof*⟩

lemma *div-natify2* [*simp*]: $m\ div\ natify(n) = m\ div\ n$
⟨*proof*⟩

27.2 Typing rules

lemma *raw-add-type*: $[\![m \in nat;\ \ n \in nat]\!] \Longrightarrow raw\text{-}add\ (m,\ n) \in nat$

⟨*proof*⟩

lemma *add-type* [*iff*, *TC*]: $m \ \#+ \ n \in nat$
⟨*proof*⟩

lemma *raw-mult-type*: $[\![m{\in}nat; \ \ n{\in}nat]\!] \Longrightarrow raw\text{-}mult\ (m,\ n) \in nat$
⟨*proof*⟩

lemma *mult-type* [*iff*, *TC*]: $m \ \#* \ n \in nat$
⟨*proof*⟩

lemma *raw-diff-type*: $[\![m{\in}nat; \ \ n{\in}nat]\!] \Longrightarrow raw\text{-}diff\ (m,\ n) \in nat$
⟨*proof*⟩

lemma *diff-type* [*iff*, *TC*]: $m \ \#- \ n \in nat$
⟨*proof*⟩

lemma *diff-0-eq-0* [*simp*]: $0 \ \#- \ n = 0$
⟨*proof*⟩

lemma *diff-succ-succ* [*simp*]: $succ(m) \ \#- \ succ(n) = m \ \#- \ n$
⟨*proof*⟩

declare *raw-diff-succ* [*simp del*]

lemma *diff-0* [*simp*]: $m \ \#- \ 0 = natify(m)$
⟨*proof*⟩

lemma *diff-le-self*: $m{\in}nat \Longrightarrow (m \ \#- \ n) \leq m$
⟨*proof*⟩

27.3 Addition

lemma *add-0-natify* [*simp*]: $0 \ \#+ \ m = natify(m)$
⟨*proof*⟩

lemma *add-succ* [*simp*]: $succ(m) \ \#+ \ n = succ(m \ \#+ \ n)$
⟨*proof*⟩

lemma *add-0*: $m \in nat \Longrightarrow 0 \ \#+ \ m = m$
⟨*proof*⟩

lemma *add-assoc*: $(m \#+ n) \#+ k = m \#+ (n \#+ k)$
⟨*proof*⟩

lemma *add-0-right-natify* [*simp*]: $m \#+ 0 = natify(m)$
⟨*proof*⟩

lemma *add-succ-right* [*simp*]: $m \#+ succ(n) = succ(m \#+ n)$
⟨*proof*⟩

lemma *add-0-right*: $m \in nat \Longrightarrow m \#+ 0 = m$
⟨*proof*⟩

lemma *add-commute*: $m \#+ n = n \#+ m$
⟨*proof*⟩

lemma *add-left-commute*: $m\#+(n\#+k)=n\#+(m\#+k)$
⟨*proof*⟩

lemmas *add-ac* = *add-assoc add-commute add-left-commute*

lemma *raw-add-left-cancel*:
$[\![raw\text{-}add(k,\ m) = raw\text{-}add(k,\ n);\ \ k{\in}nat]\!] \Longrightarrow m{=}n$
⟨*proof*⟩

lemma *add-left-cancel-natify*: $k \#+ m = k \#+ n \Longrightarrow natify(m) = natify(n)$
⟨*proof*⟩

lemma *add-left-cancel*:
$[\![i = j;\ \ i \#+ m = j \#+ n;\ \ m{\in}nat;\ \ n{\in}nat]\!] \Longrightarrow m = n$
⟨*proof*⟩

lemma *add-le-elim1-natify*: $k\#+m \le k\#+n \Longrightarrow natify(m) \le natify(n)$
⟨*proof*⟩

lemma *add-le-elim1*: $[\![k\#+m \le k\#+n;\ m \in nat;\ n \in nat]\!] \Longrightarrow m \le n$
⟨*proof*⟩

lemma *add-lt-elim1-natify*: $k\#+m < k\#+n \Longrightarrow natify(m) < natify(n)$
⟨*proof*⟩

lemma *add-lt-elim1*: $[\![k\#+m < k\#+n;\ m \in nat;\ n \in nat]\!] \Longrightarrow m < n$

⟨*proof*⟩

lemma *zero-less-add*: ⟦$n \in nat$; $m \in nat$⟧ ⟹ $0 < m \#+ n \longleftrightarrow (0<m \mid 0<n)$
⟨*proof*⟩

27.4 Monotonicity of Addition

lemma *add-lt-mono1*: ⟦$i<j$; $j \in nat$⟧ ⟹ $i\#+k < j\#+k$
⟨*proof*⟩

strict, in second argument

lemma *add-lt-mono2*: ⟦$i<j$; $j \in nat$⟧ ⟹ $k\#+i < k\#+j$
⟨*proof*⟩

A [clumsy] way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *Ord-lt-mono-imp-le-mono*:
assumes *lt-mono*: $\bigwedge i\ j.$ ⟦$i<j$; $j:k$⟧ ⟹ $f(i) < f(j)$
and *ford*: $\bigwedge i.\ i:k \Longrightarrow Ord(f(i))$
and *leij*: $i \leq j$
and *jink*: $j:k$
shows $f(i) \leq f(j)$
⟨*proof*⟩

$\leq$ monotonicity, 1st argument

lemma *add-le-mono1*: ⟦$i \leq j$; $j \in nat$⟧ ⟹ $i\#+k \leq j\#+k$
⟨*proof*⟩

$\leq$ monotonicity, both arguments

lemma *add-le-mono*: ⟦$i \leq j$; $k \leq l$; $j \in nat$; $l \in nat$⟧ ⟹ $i\#+k \leq j\#+l$
⟨*proof*⟩

Combinations of less-than and less-than-or-equals

lemma *add-lt-le-mono*: ⟦$i<j$; $k \leq l$; $j \in nat$; $l \in nat$⟧ ⟹ $i\#+k < j\#+l$
⟨*proof*⟩

lemma *add-le-lt-mono*: ⟦$i \leq j$; $k<l$; $j \in nat$; $l \in nat$⟧ ⟹ $i\#+k < j\#+l$
⟨*proof*⟩

Less-than: in other words, strict in both arguments

lemma *add-lt-mono*: ⟦$i<j$; $k<l$; $j \in nat$; $l \in nat$⟧ ⟹ $i\#+k < j\#+l$
⟨*proof*⟩

lemma *diff-add-inverse*: $(n\#+m) \#- n = natify(m)$
⟨*proof*⟩

lemma *diff-add-inverse2*: $(m\#+n) \#- n = natify(m)$

⟨*proof*⟩

lemma *diff-cancel*: $(k\#+m)\ \#-\ (k\#+n) = m\ \#-\ n$
⟨*proof*⟩

lemma *diff-cancel2*: $(m\#+k)\ \#-\ (n\#+k) = m\ \#-\ n$
⟨*proof*⟩

lemma *diff-add-0*: $n\ \#-\ (n\#+m) = 0$
⟨*proof*⟩

lemma *pred-0* [*simp*]: $pred(0) = 0$
⟨*proof*⟩

lemma *eq-succ-imp-eq-m1*: $[\![i = succ(j);\ i{\in}nat]\!] \Longrightarrow j = i\ \#-\ 1\ \wedge\ j\ {\in}nat$
⟨*proof*⟩

lemma *pred-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow pred(i \cup j) = pred(i) \cup pred(j)$
⟨*proof*⟩

lemma *pred-type* [*TC*,*simp*]:
$i \in nat \Longrightarrow pred(i) \in nat$
⟨*proof*⟩

lemma *nat-diff-pred*: $[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow i\ \#-\ succ(j) = pred(i\ \#-\ j)$
⟨*proof*⟩

lemma *diff-succ-eq-pred*: $i\ \#-\ succ(j) = pred(i\ \#-\ j)$
⟨*proof*⟩

lemma *nat-diff-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat;\ k{\in}nat]\!] \Longrightarrow (i \cup j)\ \#-\ k = (i\#-k) \cup (j\#-k)$
⟨*proof*⟩

lemma *diff-Un-distrib*:
$[\![i{\in}nat;\ j{\in}nat]\!] \Longrightarrow (i \cup j)\ \#-\ k = (i\#-k) \cup (j\#-k)$
⟨*proof*⟩

We actually prove $i\ \#-\ j\ \#-\ k = i\ \#-\ (j\ \#+\ k)$

lemma *diff-diff-left* [*simplified*]:
$natify(i)\#-natify(j)\#-k = natify(i)\ \#-\ (natify(j)\#+k)$
⟨*proof*⟩

lemma *eq-add-iff*: $(u\ \#+\ m = u\ \#+\ n) \longleftrightarrow (0\ \#+\ m = natify(n))$
⟨*proof*⟩

lemma *less-add-iff*: $(u \#+ m < u \#+ n) \longleftrightarrow (0 \#+ m < natify(n))$
⟨*proof*⟩

lemma *diff-add-eq*: $((u \#+ m) \#- (u \#+ n)) = ((0 \#+ m) \#- n)$
⟨*proof*⟩

lemma *eq-cong2*: $u = u' \Longrightarrow (t{\equiv}u) \equiv (t{\equiv}u')$
⟨*proof*⟩

lemma *iff-cong2*: $u \longleftrightarrow u' \Longrightarrow (t{\equiv}u) \equiv (t{\equiv}u')$
⟨*proof*⟩

27.5 Multiplication

lemma *mult-0* [*simp*]: $0 \#* m = 0$
⟨*proof*⟩

lemma *mult-succ* [*simp*]: $succ(m) \#* n = n \#+ (m \#* n)$
⟨*proof*⟩

lemma *mult-0-right* [*simp*]: $m \#* 0 = 0$
⟨*proof*⟩

lemma *mult-succ-right* [*simp*]: $m \#* succ(n) = m \#+ (m \#* n)$
⟨*proof*⟩

lemma *mult-1-natify* [*simp*]: $1 \#* n = natify(n)$
⟨*proof*⟩

lemma *mult-1-right-natify* [*simp*]: $n \#* 1 = natify(n)$
⟨*proof*⟩

lemma *mult-1*: $n \in nat \Longrightarrow 1 \#* n = n$
⟨*proof*⟩

lemma *mult-1-right*: $n \in nat \Longrightarrow n \#* 1 = n$
⟨*proof*⟩

lemma *mult-commute*: $m \#* n = n \#* m$
⟨*proof*⟩

lemma *add-mult-distrib*: $(m \#+ n) \#* k = (m \#* k) \#+ (n \#* k)$
⟨*proof*⟩

lemma *add-mult-distrib-left*: $k \#* (m \#+ n) = (k \#* m) \#+ (k \#* n)$
⟨*proof*⟩

lemma *mult-assoc*: $(m \#* n) \#* k = m \#* (n \#* k)$
⟨*proof*⟩

lemma *mult-left-commute*: $m \#* (n \#* k) = n \#* (m \#* k)$
⟨*proof*⟩

lemmas *mult-ac = mult-assoc mult-commute mult-left-commute*

lemma *lt-succ-eq-0-disj*:
$[\![m{\in}nat;\ n{\in}nat]\!]$
$\Longrightarrow (m < succ(n)) \longleftrightarrow (m = 0 \mid (\exists j{\in}nat.\ m = succ(j) \wedge j < n))$
⟨*proof*⟩

lemma *less-diff-conv* [*rule-format*]:
$[\![j{\in}nat;\ k{\in}nat]\!] \Longrightarrow \forall i{\in}nat.\ (i < j \#- k) \longleftrightarrow (i \#+ k < j)$
⟨*proof*⟩

lemmas *nat-typechecks = rec-type nat-0I nat-1I nat-succI Ord-nat*

end

28 Arithmetic with simplification

theory *ArithSimp*
imports *Arith*
begin

⟨*ML*⟩

28.1 Difference

lemma *diff-self-eq-0* [*simp*]: $m \#- m = 0$
⟨*proof*⟩

lemma *add-diff-inverse*: $[\![n \leq m;\ \ m{:}nat]\!] \Longrightarrow n \#+ (m\#-n) = m$
⟨*proof*⟩

lemma *add-diff-inverse2*: $[\![n \leq m;\ \ m{:}nat]\!] \Longrightarrow (m\#-n) \#+ n = m$

⟨*proof*⟩

lemma *diff-succ*: ⟦$n \leq m$; m:*nat*⟧ ⟹ *succ*(m) #− n = *succ*(m#−n)
⟨*proof*⟩

lemma *zero-less-diff* [*simp*]:
⟦m: *nat*; n: *nat*⟧ ⟹ $0 < (n \#- m) \quad \longleftrightarrow \quad m<n$
⟨*proof*⟩

lemma *diff-mult-distrib*: $(m \#- n) \#* k = (m \#* k) \#- (n \#* k)$
⟨*proof*⟩

lemma *diff-mult-distrib2*: $k \#* (m \#- n) = (k \#* m) \#- (k \#* n)$
⟨*proof*⟩

28.2 Remainder

lemma *div-termination*: ⟦$0<n$; $n \leq m$; m:*nat*⟧ ⟹ $m \#- n < m$
⟨*proof*⟩

lemmas *div-rls* =
nat-typechecks Ord-transrec-type apply-funtype
div-termination [*THEN ltD*]
nat-into-Ord not-lt-iff-le [*THEN iffD1*]

lemma *raw-mod-type*: ⟦m:*nat*; n:*nat*⟧ ⟹ *raw-mod* (m, n) ∈ *nat*
⟨*proof*⟩

lemma *mod-type* [*TC*,*iff*]: m *mod* n ∈ *nat*
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-DIV*: a *div* 0 = 0
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-MOD*: a *mod* 0 = *natify*(a)
⟨*proof*⟩

lemma *raw-mod-less*: $m<n$ ⟹ *raw-mod* (m,n) = m
⟨*proof*⟩

lemma *mod-less* [*simp*]: ⟦$m<n$; n ∈ *nat*⟧ ⟹ m *mod* n = m

⟨*proof*⟩

lemma *raw-mod-geq*:
    ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *raw-mod* (*m*, *n*) = *raw-mod* (*m*#−*n*, *n*)
⟨*proof*⟩

lemma *mod-geq*: ⟦*n* ≤ *m*; *m*:*nat*⟧ ⟹ *m mod n* = (*m*#−*n*) *mod n*
⟨*proof*⟩

28.3 Division

lemma *raw-div-type*: ⟦*m*:*nat*; *n*:*nat*⟧ ⟹ *raw-div* (*m*, *n*) ∈ *nat*
⟨*proof*⟩

lemma *div-type* [*TC*,*iff*]: *m div n* ∈ *nat*
⟨*proof*⟩

lemma *raw-div-less*: *m*<*n* ⟹ *raw-div* (*m*,*n*) = *0*
⟨*proof*⟩

lemma *div-less* [*simp*]: ⟦*m*<*n*; *n* ∈ *nat*⟧ ⟹ *m div n* = *0*
⟨*proof*⟩

lemma *raw-div-geq*: ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *raw-div*(*m*,*n*) = *succ*(*raw-div*(*m*#−*n*, *n*))
⟨*proof*⟩

lemma *div-geq* [*simp*]:
    ⟦*0*<*n*; *n* ≤ *m*; *m*:*nat*⟧ ⟹ *m div n* = *succ* ((*m*#−*n*) *div n*)
⟨*proof*⟩

declare *div-less* [*simp*] *div-geq* [*simp*]

lemma *mod-div-lemma*: ⟦*m*: *nat*; *n*: *nat*⟧ ⟹ (*m div n*)#∗*n* #+ *m mod n* = *m*
⟨*proof*⟩

lemma *mod-div-equality-natify*: (*m div n*)#∗*n* #+ *m mod n* = *natify*(*m*)
⟨*proof*⟩

lemma *mod-div-equality*: *m*: *nat* ⟹ (*m div n*)#∗*n* #+ *m mod n* = *m*
⟨*proof*⟩

28.4 Further Facts about Remainder

(mainly for mutilated chess board)

lemma *mod-succ-lemma*:

$[\![0<n;\ \ m{:}nat;\ \ n{:}nat]\!]$
$\Longrightarrow succ(m)\ mod\ n = (if\ succ(m\ mod\ n) = n\ then\ 0\ else\ succ(m\ mod\ n))$
⟨*proof*⟩

lemma *mod-succ*:
$n{:}nat \Longrightarrow succ(m)\ mod\ n = (if\ succ(m\ mod\ n) = n\ then\ 0\ else\ succ(m\ mod\ n))$
⟨*proof*⟩

lemma *mod-less-divisor*: $[\![0<n;\ \ n{:}nat]\!] \Longrightarrow m\ mod\ n < n$
⟨*proof*⟩

lemma *mod-1-eq* [*simp*]: $m\ mod\ 1 = 0$
⟨*proof*⟩

lemma *mod2-cases*: $b<2 \Longrightarrow k\ mod\ 2 = b \mid k\ mod\ 2 = (if\ b=1\ then\ 0\ else\ 1)$
⟨*proof*⟩

lemma *mod2-succ-succ* [*simp*]: $succ(succ(m))\ mod\ 2 = m\ mod\ 2$
⟨*proof*⟩

lemma *mod2-add-more* [*simp*]: $(m\#+m\#+n)\ mod\ 2 = n\ mod\ 2$
⟨*proof*⟩

lemma *mod2-add-self* [*simp*]: $(m\#+m)\ mod\ 2 = 0$
⟨*proof*⟩

28.5 Additional theorems about ≤

lemma *add-le-self*: $m{:}nat \Longrightarrow m \le (m\ \#+\ n)$
⟨*proof*⟩

lemma *add-le-self2*: $m{:}nat \Longrightarrow m \le (n\ \#+\ m)$
⟨*proof*⟩

lemma *mult-le-mono1*: $[\![i \le j;\ j{:}nat]\!] \Longrightarrow (i\#*k) \le (j\#*k)$
⟨*proof*⟩

lemma *mult-le-mono*: $[\![i \le j;\ k \le l;\ j{:}nat;\ l{:}nat]\!] \Longrightarrow i\#*k \le j\#*l$
⟨*proof*⟩

lemma *mult-lt-mono2*: $[\![i<j;\ 0<k;\ j{:}nat;\ k{:}nat]\!] \Longrightarrow k\#*i < k\#*j$
⟨*proof*⟩

lemma *mult-lt-mono1*: $[\![i<j;\ 0<k;\ j{:}nat;\ k{:}nat]\!] \Longrightarrow i\#*k < j\#*k$
⟨*proof*⟩

lemma *add-eq-0-iff* [*iff*]: $m\#+n = 0 \longleftrightarrow natify(m)=0 \wedge natify(n)=0$
⟨*proof*⟩

lemma *zero-lt-mult-iff* [*iff*]: $0 < m\#*n \longleftrightarrow 0 < natify(m) \wedge 0 < natify(n)$
⟨*proof*⟩

lemma *mult-eq-1-iff* [*iff*]: $m\#*n = 1 \longleftrightarrow natify(m)=1 \wedge natify(n)=1$
⟨*proof*⟩

lemma *mult-is-zero*: $[\![m\text{: } nat;\ n\text{: } nat]\!] \Longrightarrow (m \#* n = 0) \longleftrightarrow (m = 0 \mid n = 0)$
⟨*proof*⟩

lemma *mult-is-zero-natify* [*iff*]:
$(m \#* n = 0) \longleftrightarrow (natify(m) = 0 \mid natify(n) = 0)$
⟨*proof*⟩

28.6 Cancellation Laws for Common Factors in Comparisons

lemma *mult-less-cancel-lemma*:
$[\![k\text{: } nat;\ m\text{: } nat;\ n\text{: } nat]\!] \Longrightarrow (m\#*k < n\#*k) \longleftrightarrow (0<k \wedge m<n)$
⟨*proof*⟩

lemma *mult-less-cancel2* [*simp*]:
$(m\#*k < n\#*k) \longleftrightarrow (0 < natify(k) \wedge natify(m) < natify(n))$
⟨*proof*⟩

lemma *mult-less-cancel1* [*simp*]:
$(k\#*m < k\#*n) \longleftrightarrow (0 < natify(k) \wedge natify(m) < natify(n))$
⟨*proof*⟩

lemma *mult-le-cancel2* [*simp*]: $(m\#*k \leq n\#*k) \longleftrightarrow (0 < natify(k) \longrightarrow natify(m) \leq natify(n))$
⟨*proof*⟩

lemma *mult-le-cancel1* [*simp*]: $(k\#*m \leq k\#*n) \longleftrightarrow (0 < natify(k) \longrightarrow natify(m) \leq natify(n))$
⟨*proof*⟩

lemma *mult-le-cancel-le1*: $k \in nat \Longrightarrow k \#* m \leq k \longleftrightarrow (0 < k \longrightarrow natify(m) \leq 1)$
⟨*proof*⟩

lemma *Ord-eq-iff-le*: $[\![Ord(m);\ Ord(n)]\!] \Longrightarrow m=n \longleftrightarrow (m \leq n \wedge n \leq m)$
⟨*proof*⟩

lemma *mult-cancel2-lemma*:
$[\![k\text{: } nat;\ m\text{: } nat;\ n\text{: } nat]\!] \Longrightarrow (m\#*k = n\#*k) \longleftrightarrow (m=n \mid k=0)$

⟨*proof*⟩

lemma *mult-cancel2* [*simp*]:
(*m*#∗*k* = *n*#∗*k*) ⟷ (*natify*(*m*) = *natify*(*n*) | *natify*(*k*) = *0*)
⟨*proof*⟩

lemma *mult-cancel1* [*simp*]:
(*k*#∗*m* = *k*#∗*n*) ⟷ (*natify*(*m*) = *natify*(*n*) | *natify*(*k*) = *0*)
⟨*proof*⟩

lemma *div-cancel-raw*:
⟦*0*<*n*; *0*<*k*; *k*:*nat*; *m*:*nat*; *n*:*nat*⟧ ⟹ (*k*#∗*m*) *div* (*k*#∗*n*) = *m div n*
⟨*proof*⟩

lemma *div-cancel*:
⟦*0* < *natify*(*n*); *0* < *natify*(*k*)⟧ ⟹ (*k*#∗*m*) *div* (*k*#∗*n*) = *m div n*
⟨*proof*⟩

28.7 More Lemmas about Remainder

lemma *mult-mod-distrib-raw*:
⟦*k*:*nat*; *m*:*nat*; *n*:*nat*⟧ ⟹ (*k*#∗*m*) *mod* (*k*#∗*n*) = *k* #∗ (*m mod n*)
⟨*proof*⟩

lemma *mod-mult-distrib2*: *k* #∗ (*m mod n*) = (*k*#∗*m*) *mod* (*k*#∗*n*)
⟨*proof*⟩

lemma *mult-mod-distrib*: (*m mod n*) #∗ *k* = (*m*#∗*k*) *mod* (*n*#∗*k*)
⟨*proof*⟩

lemma *mod-add-self2-raw*: *n* ∈ *nat* ⟹ (*m* #+ *n*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *mod-add-self2* [*simp*]: (*m* #+ *n*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *mod-add-self1* [*simp*]: (*n*#+*m*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *mod-mult-self1-raw*: *k* ∈ *nat* ⟹ (*m* #+ *k*#∗*n*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *mod-mult-self1* [*simp*]: (*m* #+ *k*#∗*n*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *mod-mult-self2* [*simp*]: (*m* #+ *n*#∗*k*) *mod n* = *m mod n*

⟨*proof*⟩

lemma *mult-eq-self-implies-10*: $m = m\#*n \Longrightarrow natify(n)=1 \mid m=0$
⟨*proof*⟩

lemma *less-imp-succ-add* [*rule-format*]:
$[\![m<n;\ n{:}\ nat]\!] \Longrightarrow \exists\, k{\in}nat.\ n = succ(m\#+k)$
⟨*proof*⟩

lemma *less-iff-succ-add*:
$[\![m{:}\ nat;\ n{:}\ nat]\!] \Longrightarrow (m<n) \longleftrightarrow (\exists\, k{\in}nat.\ n = succ(m\#+k))$
⟨*proof*⟩

lemma *add-lt-elim2*:
$[\![a\ \#+\ d = b\ \#+\ c;\ a < b;\ b \in nat;\ c \in nat;\ d \in nat]\!] \Longrightarrow c < d$
⟨*proof*⟩

lemma *add-le-elim2*:
$[\![a\ \#+\ d = b\ \#+\ c;\ a \le b;\ b \in nat;\ c \in nat;\ d \in nat]\!] \Longrightarrow c \le d$
⟨*proof*⟩

28.7.1 More Lemmas About Difference

lemma *diff-is-0-lemma*:
$[\![m{:}\ nat;\ n{:}\ nat]\!] \Longrightarrow m\ \#-\ n = 0 \longleftrightarrow m \le n$
⟨*proof*⟩

lemma *diff-is-0-iff*: $m\ \#-\ n = 0 \longleftrightarrow natify(m) \le natify(n)$
⟨*proof*⟩

lemma *nat-lt-imp-diff-eq-0*:
$[\![a{:}nat;\ b{:}nat;\ a<b]\!] \Longrightarrow a\ \#-\ b = 0$
⟨*proof*⟩

lemma *raw-nat-diff-split*:
$[\![a{:}nat;\ b{:}nat]\!] \Longrightarrow$
$(P(a\ \#-\ b)) \longleftrightarrow ((a < b \longrightarrow P(0)) \wedge (\forall\, d{\in}nat.\ a = b\ \#+\ d \longrightarrow P(d)))$
⟨*proof*⟩

lemma *nat-diff-split*:
$(P(a\ \#-\ b)) \longleftrightarrow$
$(natify(a) < natify(b) \longrightarrow P(0)) \wedge (\forall\, d{\in}nat.\ natify(a) = b\ \#+\ d \longrightarrow P(d))$
⟨*proof*⟩

Difference and less-than

lemma *diff-lt-imp-lt*: $[\![(k\#-i) < (k\#-j);\ i{\in}nat;\ j{\in}nat;\ k{\in}nat]\!] \Longrightarrow j<i$
⟨*proof*⟩

lemma *lt-imp-diff-lt*: $[\![j<i;\ i\le k;\ k\in nat]\!] \Longrightarrow (k\#-i) < (k\#-j)$
⟨*proof*⟩

lemma *diff-lt-iff-lt*: $[\![i\le k;\ j\in nat;\ k\in nat]\!] \Longrightarrow (k\#-i) < (k\#-j) \longleftrightarrow j<i$
⟨*proof*⟩

end

29 Lists in Zermelo-Fraenkel Set Theory

theory *List* **imports** *Datatype ArithSimp* **begin**

consts
list :: $i \Rightarrow i$

datatype
$list(A) = Nil \mid Cons\ (a \in A,\ l \in list(A))$

syntax
-Nil :: *i* (‹[]›)
-List :: *is* ⇒ *i* (‹[(-)]›)

translations
$[x,\ xs]$ == $CONST\ Cons(x,\ [xs])$
$[x]$ == $CONST\ Cons(x,\ [])$
$[]$ == $CONST\ Nil$

consts
length :: $i \Rightarrow i$
hd :: $i \Rightarrow i$
tl :: $i \Rightarrow i$

primrec
$length([]) = 0$
$length(Cons(a,l)) = succ(length(l))$

primrec
$hd([]) = 0$
$hd(Cons(a,l)) = a$

primrec
$tl([]) = []$
$tl(Cons(a,l)) = l$

consts

map :: $[i \Rightarrow i,\ i] \Rightarrow i$
set-of-list :: $i \Rightarrow i$
app :: $[i,i] \Rightarrow i$ (**infixr** ‹@› *60*)

primrec
$map(f,[]) = []$
$map(f,Cons(a,l)) = Cons(f(a),\ map(f,l))$

primrec
$set\text{-}of\text{-}list([]) = 0$
$set\text{-}of\text{-}list(Cons(a,l)) = cons(a,\ set\text{-}of\text{-}list(l))$

primrec
app-Nil: $[] \ @\ ys = ys$
app-Cons: $(Cons(a,l)) \ @\ ys = Cons(a,\ l \ @\ ys)$

consts
rev :: $i \Rightarrow i$
flat :: $i \Rightarrow i$
list-add :: $i \Rightarrow i$

primrec
$rev([]) = []$
$rev(Cons(a,l)) = rev(l) \ @\ [a]$

primrec
$flat([]) = []$
$flat(Cons(l,ls)) = l \ @\ flat(ls)$

primrec
$list\text{-}add([]) = 0$
$list\text{-}add(Cons(a,l)) = a \ \#+\ list\text{-}add(l)$

consts
drop :: $[i,i] \Rightarrow i$

primrec
drop-0: $drop(0,l) = l$
drop-succ: $drop(succ(i),\ l) = tl\ (drop(i,l))$

definition

take :: $[i,i] \Rightarrow i$ **where**
$take(n,\ as) \equiv list\text{-}rec(\lambda n{\in}nat.\ [],$

$\lambda a\ l\ r.\ \lambda n{\in}nat.\ nat\text{-}case([],\ \lambda m.\ Cons(a,\ r`m),\ n),\ as)`n$

definition
nth :: $[i,\ i]{\Rightarrow}i$ **where**
— returns the (n+1)th element of a list, or 0 if the list is too short.
$nth(n,\ as) \equiv list\text{-}rec(\lambda n{\in}nat.\ 0,$
$\lambda a\ l\ r.\ \lambda n{\in}nat.\ nat\text{-}case(a,\ \lambda m.\ r`m,\ n),\ as)\ `\ n$

definition
list-update :: $[i,\ i,\ i]{\Rightarrow}i$ **where**
$list\text{-}update(xs,\ i,\ v) \equiv list\text{-}rec(\lambda n{\in}nat.\ Nil,$
$\lambda u\ us\ vs.\ \lambda n{\in}nat.\ nat\text{-}case(Cons(v,\ us),\ \lambda m.\ Cons(u,\ vs`m),\ n),\ xs)`i$

consts
filter :: $[i{\Rightarrow}o,\ i] \Rightarrow i$
upt :: $[i,\ i] \Rightarrow i$

primrec
$filter(P,\ Nil) = Nil$
$filter(P,\ Cons(x,\ xs)) =$
$(if\ P(x)\ then\ Cons(x,\ filter(P,\ xs))\ else\ filter(P,\ xs))$

primrec
$upt(i,\ 0) = Nil$
$upt(i,\ succ(j)) = (if\ i \le j\ then\ upt(i,\ j)@[j]\ else\ Nil)$

definition
min :: $[i,i] \Rightarrow i$ **where**
$min(x,\ y) \equiv (if\ x \le y\ then\ x\ else\ y)$

definition
max :: $[i,\ i] \Rightarrow i$ **where**
$max(x,\ y) \equiv (if\ x \le y\ then\ y\ else\ x)$

declare *list.intros* [*simp*,*TC*]

inductive-cases *ConsE*: $Cons(a,l) \in list(A)$

lemma *Cons-type-iff* [*simp*]: $Cons(a,l) \in list(A) \longleftrightarrow a \in A \wedge l \in list(A)$
⟨*proof*⟩

lemma *Cons-iff*: $Cons(a,l)=Cons(a',l') \longleftrightarrow a=a' \wedge l=l'$
⟨*proof*⟩

lemma *Nil-Cons-iff*: $\neg\ Nil=Cons(a,l)$

⟨*proof*⟩

lemma *list-unfold*: $list(A) = \{0\} + (A * list(A))$
⟨*proof*⟩

lemma *list-mono*: $A<=B \Longrightarrow list(A) \subseteq list(B)$
⟨*proof*⟩

lemma *list-univ*: $list(univ(A)) \subseteq univ(A)$
⟨*proof*⟩

lemmas *list-subset-univ* = *subset-trans* [*OF list-mono list-univ*]

lemma *list-into-univ*: $[\![l \in list(A);\ A \subseteq univ(B)]\!] \Longrightarrow l \in univ(B)$
⟨*proof*⟩

lemma *list-case-type*:
$[\![l \in list(A);$
$c \in C(Nil);$
$\bigwedge x\ y.\ [\![x \in A;\ y \in list(A)]\!] \Longrightarrow h(x,y): C(Cons(x,y))$
$]\!] \Longrightarrow list\text{-}case(c,h,l) \in C(l)$
⟨*proof*⟩

lemma *list-0-triv*: $list(0) = \{Nil\}$
⟨*proof*⟩

lemma *tl-type*: $l \in list(A) \Longrightarrow tl(l) \in list(A)$
⟨*proof*⟩

lemma *drop-Nil* [*simp*]: $i \in nat \Longrightarrow drop(i,\ Nil) = Nil$
⟨*proof*⟩

lemma *drop-succ-Cons* [*simp*]: $i \in nat \Longrightarrow drop(succ(i),\ Cons(a,l)) = drop(i,l)$
⟨*proof*⟩

lemma *drop-type* [*simp*,*TC*]: $[\![i \in nat;\ l \in list(A)]\!] \Longrightarrow drop(i,l) \in list(A)$
⟨*proof*⟩

declare *drop-succ* [*simp del*]

lemma *list-rec-type* [*TC*]:
$[\![l \in list(A);$
$c \in C(Nil);$
$\bigwedge x\ y\ r.\ [\![x \in A;\ \ y \in list(A);\ \ r \in C(y)]\!] \Longrightarrow h(x,y,r) : C(Cons(x,y))$
$]\!] \Longrightarrow list\text{-}rec(c,h,l) \in C(l)$
⟨*proof*⟩

lemma *map-type* [*TC*]:
$[\![l \in list(A);\ \ \bigwedge x.\ x \in A \Longrightarrow h(x) : B]\!] \Longrightarrow map(h,l) \in list(B)$
⟨*proof*⟩

lemma *map-type2* [*TC*]: $l \in list(A) \Longrightarrow map(h,l) \in list(\{h(u).\ u \in A\})$
⟨*proof*⟩

lemma *length-type* [*TC*]: $l \in list(A) \Longrightarrow length(l) \in nat$
⟨*proof*⟩

lemma *lt-length-in-nat*:
$[\![x < length(xs);\ xs \in list(A)]\!] \Longrightarrow x \in nat$
⟨*proof*⟩

lemma *app-type* [*TC*]: $[\![xs : list(A);\ \ ys : list(A)]\!] \Longrightarrow xs@ys \in list(A)$
⟨*proof*⟩

lemma *rev-type* [*TC*]: $xs : list(A) \Longrightarrow rev(xs) \in list(A)$
⟨*proof*⟩

lemma *flat-type* [*TC*]: $ls : list(list(A)) \Longrightarrow flat(ls) \in list(A)$
⟨*proof*⟩

lemma *set-of-list-type* [*TC*]: $l \in list(A) \Longrightarrow set\text{-}of\text{-}list(l) \in Pow(A)$

⟨*proof*⟩

lemma *set-of-list-append*:
xs: *list*(*A*) ⟹ *set-of-list* (*xs*@*ys*) = *set-of-list*(*xs*) ∪ *set-of-list*(*ys*)
⟨*proof*⟩

lemma *list-add-type* [*TC*]: *xs*: *list*(*nat*) ⟹ *list-add*(*xs*) ∈ *nat*
⟨*proof*⟩

lemma *map-ident* [*simp*]: *l* ∈ *list*(*A*) ⟹ *map*(λ*u*. *u*, *l*) = *l*
⟨*proof*⟩

lemma *map-compose*: *l* ∈ *list*(*A*) ⟹ *map*(*h*, *map*(*j*,*l*)) = *map*(λ*u*. *h*(*j*(*u*)), *l*)
⟨*proof*⟩

lemma *map-app-distrib*: *xs*: *list*(*A*) ⟹ *map*(*h*, *xs*@*ys*) = *map*(*h*,*xs*) @ *map*(*h*,*ys*)
⟨*proof*⟩

lemma *map-flat*: *ls*: *list*(*list*(*A*)) ⟹ *map*(*h*, *flat*(*ls*)) = *flat*(*map*(*map*(*h*),*ls*))
⟨*proof*⟩

lemma *list-rec-map*:
l ∈ *list*(*A*) ⟹
list-rec(*c*, *d*, *map*(*h*,*l*)) =
list-rec(*c*, λ*x xs r*. *d*(*h*(*x*), *map*(*h*,*xs*), *r*), *l*)
⟨*proof*⟩

lemmas *list-CollectD* = *Collect-subset* [*THEN list-mono*, *THEN subsetD*]

lemma *map-list-Collect*: *l* ∈ *list*({*x* ∈ *A*. *h*(*x*)=*j*(*x*)}) ⟹ *map*(*h*,*l*) = *map*(*j*,*l*)
⟨*proof*⟩

lemma *length-map* [*simp*]: *xs*: *list*(*A*) ⟹ *length*(*map*(*h*,*xs*)) = *length*(*xs*)
⟨*proof*⟩

lemma *length-app* [*simp*]:
⟦*xs*: *list*(*A*); *ys*: *list*(*A*)⟧
⟹ *length*(*xs*@*ys*) = *length*(*xs*) #+ *length*(*ys*)

⟨*proof*⟩

lemma *length-rev* [*simp*]: *xs*: *list*(*A*) ⟹ *length*(*rev*(*xs*)) = *length*(*xs*)
⟨*proof*⟩

lemma *length-flat*:
  ls: *list*(*list*(*A*)) ⟹ *length*(*flat*(*ls*)) = *list-add*(*map*(*length*,*ls*))
⟨*proof*⟩

lemma *drop-length-Cons* [*rule-format*]:
  xs: *list*(*A*) ⟹
    ∀ *x*. ∃ *z zs*. *drop*(*length*(*xs*), *Cons*(*x*,*xs*)) = *Cons*(*z*,*zs*)
⟨*proof*⟩

lemma *drop-length* [*rule-format*]:
  l ∈ *list*(*A*) ⟹ ∀ *i* ∈ *length*(*l*). (∃ *z zs*. *drop*(*i*,*l*) = *Cons*(*z*,*zs*))
⟨*proof*⟩

lemma *app-right-Nil* [*simp*]: *xs*: *list*(*A*) ⟹ *xs*@*Nil*=*xs*
⟨*proof*⟩

lemma *app-assoc*: *xs*: *list*(*A*) ⟹ (*xs*@*ys*)@*zs* = *xs*@(*ys*@*zs*)
⟨*proof*⟩

lemma *flat-app-distrib*: *ls*: *list*(*list*(*A*)) ⟹ *flat*(*ls*@*ms*) = *flat*(*ls*)@*flat*(*ms*)
⟨*proof*⟩

lemma *rev-map-distrib*: *l* ∈ *list*(*A*) ⟹ *rev*(*map*(*h*,*l*)) = *map*(*h*,*rev*(*l*))
⟨*proof*⟩

lemma *rev-app-distrib*:
  ⟦*xs*: *list*(*A*); *ys*: *list*(*A*)⟧ ⟹ *rev*(*xs*@*ys*) = *rev*(*ys*)@*rev*(*xs*)
⟨*proof*⟩

lemma *rev-rev-ident* [*simp*]: *l* ∈ *list*(*A*) ⟹ *rev*(*rev*(*l*))=*l*
⟨*proof*⟩

lemma *rev-flat*: *ls*: *list*(*list*(*A*)) ⟹ *rev*(*flat*(*ls*)) = *flat*(*map*(*rev*,*rev*(*ls*)))
⟨*proof*⟩

lemma *list-add-app*:
$[\![xs\colon list(nat);\ \ ys\colon list(nat)]\!]$
$\Longrightarrow list\text{-}add(xs@ys) = list\text{-}add(ys)\ \#+\ list\text{-}add(xs)$
⟨*proof*⟩

lemma *list-add-rev*: $l \in list(nat) \Longrightarrow list\text{-}add(rev(l)) = list\text{-}add(l)$
⟨*proof*⟩

lemma *list-add-flat*:
$ls\colon list(list(nat)) \Longrightarrow list\text{-}add(flat(ls)) = list\text{-}add(map(list\text{-}add,ls))$
⟨*proof*⟩

lemma *list-append-induct* [*case-names Nil snoc*, *consumes 1*]:
$[\![l \in list(A);$
$P(Nil);$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in list(A);\ \ P(y)]\!] \Longrightarrow P(y\ @\ [x])$
$]\!] \Longrightarrow P(l)$
⟨*proof*⟩

lemma *list-complete-induct-lemma* [*rule-format*]:
assumes *ih*:
$\bigwedge l.\ [\![l \in list(A);$
$\forall\, l' \in list(A).\ length(l') < length(l) \longrightarrow P(l')]\!]$
$\Longrightarrow P(l)$
shows $n \in nat \Longrightarrow \forall\, l \in list(A).\ length(l) < n \longrightarrow P(l)$
⟨*proof*⟩

theorem *list-complete-induct*:
$[\![l \in list(A);$
$\bigwedge l.\ [\![l \in list(A);$
$\forall\, l' \in list(A).\ length(l') < length(l) \longrightarrow P(l')]\!]$
$\Longrightarrow P(l)$
$]\!] \Longrightarrow P(l)$
⟨*proof*⟩

lemma *min-sym*: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow min(i,j)=min(j,i)$
⟨*proof*⟩

lemma *min-type* [*simp*,*TC*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow min(i,j){:}nat$

⟨*proof*⟩

lemma *min-0* [*simp*]: $i \in nat \Longrightarrow min(0,i) = 0$
⟨*proof*⟩

lemma *min-02* [*simp*]: $i \in nat \Longrightarrow min(i, 0) = 0$
⟨*proof*⟩

lemma *lt-min-iff*: $[\![i \in nat; j \in nat; k \in nat]\!] \Longrightarrow i<min(j,k) \longleftrightarrow i<j \wedge i<k$
⟨*proof*⟩

lemma *min-succ-succ* [*simp*]:
$[\![i \in nat; j \in nat]\!] \Longrightarrow min(succ(i), succ(j))= succ(min(i, j))$
⟨*proof*⟩

lemma *filter-append* [*simp*]:
$xs{:}list(A) \Longrightarrow filter(P, xs@ys) = filter(P, xs) @ filter(P, ys)$
⟨*proof*⟩

lemma *filter-type* [*simp*,*TC*]: $xs{:}list(A) \Longrightarrow filter(P, xs){:}list(A)$
⟨*proof*⟩

lemma *length-filter*: $xs{:}list(A) \Longrightarrow length(filter(P, xs)) \leq length(xs)$
⟨*proof*⟩

lemma *filter-is-subset*: $xs{:}list(A) \Longrightarrow set\text{-}of\text{-}list(filter(P,xs)) \subseteq set\text{-}of\text{-}list(xs)$
⟨*proof*⟩

lemma *filter-False* [*simp*]: $xs{:}list(A) \Longrightarrow filter(\lambda p.\ False, xs) = Nil$
⟨*proof*⟩

lemma *filter-True* [*simp*]: $xs{:}list(A) \Longrightarrow filter(\lambda p.\ True, xs) = xs$
⟨*proof*⟩

lemma *length-is-0-iff* [*simp*]: $xs{:}list(A) \Longrightarrow length(xs)=0 \longleftrightarrow xs=Nil$
⟨*proof*⟩

lemma *length-is-0-iff2* [*simp*]: $xs{:}list(A) \Longrightarrow 0 = length(xs) \longleftrightarrow xs=Nil$
⟨*proof*⟩

lemma *length-tl* [*simp*]: $xs{:}list(A) \Longrightarrow length(tl(xs)) = length(xs) \#- 1$
⟨*proof*⟩

lemma *length-greater-0-iff*: $xs{:}list(A) \Longrightarrow 0{<}length(xs) \longleftrightarrow xs \neq Nil$
⟨*proof*⟩

lemma *length-succ-iff*: $xs{:}list(A) \Longrightarrow length(xs){=}succ(n) \longleftrightarrow (\exists\, y\ ys.\ xs{=}Cons(y, ys) \wedge length(ys){=}n)$
⟨*proof*⟩

lemma *append-is-Nil-iff* [*simp*]:
$xs{:}list(A) \Longrightarrow (xs@ys = Nil) \longleftrightarrow (xs{=}Nil \wedge ys = Nil)$
⟨*proof*⟩

lemma *append-is-Nil-iff2* [*simp*]:
$xs{:}list(A) \Longrightarrow (Nil = xs@ys) \longleftrightarrow (xs{=}Nil \wedge ys = Nil)$
⟨*proof*⟩

lemma *append-left-is-self-iff* [*simp*]:
$xs{:}list(A) \Longrightarrow (xs@ys = xs) \longleftrightarrow (ys = Nil)$
⟨*proof*⟩

lemma *append-left-is-self-iff2* [*simp*]:
$xs{:}list(A) \Longrightarrow (xs = xs@ys) \longleftrightarrow (ys = Nil)$
⟨*proof*⟩

lemma *append-left-is-Nil-iff* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow$
$length(ys){=}length(zs) \longrightarrow (xs@ys{=}zs \longleftrightarrow (xs{=}Nil \wedge ys{=}zs))$
⟨*proof*⟩

lemma *append-left-is-Nil-iff2* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow$
$length(ys){=}length(zs) \longrightarrow (zs{=}ys@xs \longleftrightarrow (xs{=}Nil \wedge ys{=}zs))$
⟨*proof*⟩

lemma *append-eq-append-iff* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, ys \in list(A).$
$length(xs){=}length(ys) \longrightarrow (xs@us = ys@vs) \longleftrightarrow (xs{=}ys \wedge us{=}vs)$
⟨*proof*⟩
declare *append-eq-append-iff* [*simp*]

lemma *append-eq-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, ys \in list(A).\ \forall\, us \in list(A).\ \forall\, vs \in list(A).$
$length(us) = length(vs) \longrightarrow (xs@us = ys@vs) \longrightarrow (xs{=}ys \wedge us{=}vs)$
⟨*proof*⟩

lemma *append-eq-append-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ us{:}list(A);\ vs{:}list(A);\ length(us){=}length(vs)]\!]$
$\Longrightarrow\ xs@us = ys@vs \longleftrightarrow (xs{=}ys \wedge us{=}vs)$
⟨*proof*⟩

lemma *append-self-iff* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow xs@ys{=}xs@zs \longleftrightarrow ys{=}zs$
⟨*proof*⟩

lemma *append-self-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ zs{:}list(A)]\!] \Longrightarrow ys@xs{=}zs@xs \longleftrightarrow ys{=}zs$
⟨*proof*⟩

lemma *append1-eq-iff* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, ys \in list(A).\ xs@[x] = ys@[y] \longleftrightarrow (xs = ys \wedge x{=}y)$
⟨*proof*⟩
declare *append1-eq-iff* [*simp*]

lemma *append-right-is-self-iff* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A)]\!] \Longrightarrow (xs@ys = ys) \longleftrightarrow (xs{=}Nil)$
⟨*proof*⟩

lemma *append-right-is-self-iff2* [*simp*]:
$[\![xs{:}list(A);\ ys{:}list(A)]\!] \Longrightarrow (ys = xs@ys) \longleftrightarrow (xs{=}Nil)$
⟨*proof*⟩

lemma *hd-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow xs \neq Nil \longrightarrow hd(xs\ @\ ys) = hd(xs)$
⟨*proof*⟩
declare *hd-append* [*simp*]

lemma *tl-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow xs{\neq}Nil \longrightarrow tl(xs\ @\ ys) = tl(xs)@ys$
⟨*proof*⟩
declare *tl-append* [*simp*]

lemma *rev-is-Nil-iff* [*simp*]: $xs{:}list(A) \Longrightarrow (rev(xs) = Nil \longleftrightarrow xs = Nil)$
⟨*proof*⟩

lemma *Nil-is-rev-iff* [*simp*]: $xs{:}list(A) \Longrightarrow (Nil = rev(xs) \longleftrightarrow xs = Nil)$
⟨*proof*⟩

lemma *rev-is-rev-iff* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, ys \in list(A).\ rev(xs){=}rev(ys) \longleftrightarrow xs{=}ys$
⟨*proof*⟩
declare *rev-is-rev-iff* [*simp*]

lemma *rev-list-elim* [*rule-format*]:
xs:*list*(A) $\Longrightarrow$
(xs=*Nil* $\longrightarrow$ P) $\longrightarrow$ ($\forall\, ys \in list(A).\ \forall\, y \in A.\ xs = ys@[y] \longrightarrow P) \longrightarrow P$
⟨*proof*⟩

lemma *length-drop* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, xs \in list(A).\ length(drop(n,\ xs)) = length(xs)\ \#-\ n$
⟨*proof*⟩
declare *length-drop* [*simp*]

lemma *drop-all* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, xs \in list(A).\ length(xs) \leq n \longrightarrow drop(n,\ xs)=Nil$
⟨*proof*⟩
declare *drop-all* [*simp*]

lemma *drop-append* [*rule-format*]:
$n \in nat \Longrightarrow$
$\forall\, xs \in list(A).\ drop(n,\ xs@ys) = drop(n,xs)\ @\ drop(n\ \#-\ length(xs),\ ys)$
⟨*proof*⟩

lemma *drop-drop*:
$m \in nat \Longrightarrow \forall\, xs \in list(A).\ \forall\, n \in nat.\ drop(n,\ drop(m,\ xs))=drop(n\ \#+\ m,\ xs)$
⟨*proof*⟩

lemma *take-0* [*simp*]: xs:*list*(A) $\Longrightarrow$ $take(0,\ xs) =\ Nil$
⟨*proof*⟩

lemma *take-succ-Cons* [*simp*]:
$n \in nat \Longrightarrow take(succ(n),\ Cons(a,\ xs)) = Cons(a,\ take(n,\ xs))$
⟨*proof*⟩

lemma *take-Nil* [*simp*]: $n \in nat \Longrightarrow take(n,\ Nil) = Nil$
⟨*proof*⟩

lemma *take-all* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, xs \in list(A).\ length(xs) \leq n\ \longrightarrow take(n,\ xs) = xs$
⟨*proof*⟩
declare *take-all* [*simp*]

lemma *take-type* [*rule-format*]:
xs:*list*(A) $\Longrightarrow \forall\, n \in nat.\ take(n,\ xs)$:*list*($A$)
⟨*proof*⟩
declare *take-type* [*simp*,*TC*]

lemma *take-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, ys \in list(A).\ \forall\, n \in nat.\ take(n,\ xs\ @\ ys) =$
$take(n,\ xs)\ @\ take(n\ \#-\ length(xs),\ ys)$
⟨*proof*⟩
declare *take-append* [*simp*]

lemma *take-take* [*rule-format*]:
$m \in nat \Longrightarrow$
$\forall\, xs \in list(A).\ \forall\, n \in nat.\ take(n,\ take(m,xs)) = take(min(n,\ m),\ xs)$
⟨*proof*⟩

lemma *nth-0* [*simp*]: $nth(0,\ Cons(a,\ l)) = a$
⟨*proof*⟩

lemma *nth-Cons* [*simp*]: $n \in nat \Longrightarrow nth(succ(n),\ Cons(a,l)) = nth(n,l)$
⟨*proof*⟩

lemma *nth-empty* [*simp*]: $nth(n,\ Nil) = 0$
⟨*proof*⟩

lemma *nth-type* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, n.\ n < length(xs) \longrightarrow nth(n,xs) \in A$
⟨*proof*⟩
declare *nth-type* [*simp*,*TC*]

lemma *nth-eq-0* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, n \in nat.\ length(xs) \leq n \longrightarrow nth(n,xs) = 0$
⟨*proof*⟩

lemma *nth-append* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, n \in nat.\ nth(n,\ xs\ @\ ys) = (if\ n < length(xs)\ then\ nth(n,xs)$
$else\ nth(n\ \#-\ length(xs),\ ys))$
⟨*proof*⟩

lemma *set-of-list-conv-nth*:
$xs{:}list(A)$
$\Longrightarrow set\text{-}of\text{-}list(xs) = \{x \in A.\ \exists\, i{\in}nat.\ i{<}length(xs) \wedge x = nth(i,xs)\}$
⟨*proof*⟩

lemma *nth-take-lemma* [*rule-format*]:
$k \in nat \Longrightarrow$
$\forall\, xs \in list(A).\ (\forall\, ys \in list(A).\ k \leq length(xs) \longrightarrow k \leq length(ys) \longrightarrow$

$(\forall\, i \in nat.\ i<k \longrightarrow nth(i,xs) = nth(i,ys)) \longrightarrow take(k,xs) = take(k,ys))$
⟨*proof*⟩

lemma *nth-equalityI* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ length(xs) = length(ys);$
$\forall\, i \in nat.\ i < length(xs) \longrightarrow nth(i,xs) = nth(i,ys)]\!]$
$\Longrightarrow xs = ys$
⟨*proof*⟩

lemma *take-equalityI* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(A);\ (\forall\, i \in nat.\ take(i,\ xs) = take(i,ys))]\!]$
$\Longrightarrow xs = ys$
⟨*proof*⟩

lemma *nth-drop* [*rule-format*]:
$n \in nat \Longrightarrow \forall\, i \in nat.\ \forall\, xs \in list(A).\ nth(i,\ drop(n,\ xs)) = nth(n\ \#+\ i,\ xs)$
⟨*proof*⟩

lemma *take-succ* [*rule-format*]:
$xs{\in}list(A)$
$\Longrightarrow \forall\, i.\ i < length(xs) \longrightarrow take(succ(i),\ xs) = take(i,xs)\ @\ [nth(i,\ xs)]$
⟨*proof*⟩

lemma *take-add* [*rule-format*]:
$[\![xs{\in}list(A);\ j{\in}nat]\!]$
$\Longrightarrow \forall\, i{\in}nat.\ take(i\ \#+\ j,\ xs) = take(i,xs)\ @\ take(j,\ drop(i,xs))$
⟨*proof*⟩

lemma *length-take*:
$l{\in}list(A) \Longrightarrow \forall\, n{\in}nat.\ length(take(n,l)) = min(n,\ length(l))$
⟨*proof*⟩

29.1 The function zip

Crafty definition to eliminate a type argument

consts
zip-aux :: $[i,i]{\Rightarrow}i$

primrec
zip-aux$(B,[]) =$
$(\lambda ys \in list(B).\ list\text{-}case([],\ \lambda y\ l.\ [],\ ys))$

zip-aux$(B,Cons(x,l)) =$
$(\lambda ys \in list(B).$
$list\text{-}case(Nil,\ \lambda y\ zs.\ Cons(\langle x,y\rangle,\ zip\text{-}aux(B,l)\text{`}zs),\ ys))$

definition

zip :: [*i*, *i*]⇒*i* **where**
zip(*xs*, *ys*) ≡ *zip-aux*(*set-of-list*(*ys*),*xs*) ‘*ys*

lemma *list-on-set-of-list*: *xs* ∈ *list*(*A*) ⟹ *xs* ∈ *list*(*set-of-list*(*xs*))
⟨*proof*⟩

lemma *zip-Nil* [*simp*]: *ys*:*list*(*A*) ⟹ *zip*(*Nil*, *ys*)=*Nil*
⟨*proof*⟩

lemma *zip-Nil2* [*simp*]: *xs*:*list*(*A*) ⟹ *zip*(*xs*, *Nil*)=*Nil*
⟨*proof*⟩

lemma *zip-aux-unique* [*rule-format*]:
⟦*B*<=*C*; *xs* ∈ *list*(*A*)⟧
⟹ ∀ *ys* ∈ *list*(*B*). *zip-aux*(*C*,*xs*) ‘ *ys* = *zip-aux*(*B*,*xs*) ‘ *ys*
⟨*proof*⟩

lemma *zip-Cons-Cons* [*simp*]:
⟦*xs*:*list*(*A*); *ys*:*list*(*B*); *x* ∈ *A*; *y* ∈ *B*⟧ ⟹
zip(*Cons*(*x*,*xs*), *Cons*(*y*, *ys*)) = *Cons*(⟨*x*,*y*⟩, *zip*(*xs*, *ys*))
⟨*proof*⟩

lemma *zip-type* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *ys* ∈ *list*(*B*). *zip*(*xs*, *ys*):*list*(*A*∗*B*)
⟨*proof*⟩
declare *zip-type* [*simp*,*TC*]

lemma *length-zip* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *ys* ∈ *list*(*B*). *length*(*zip*(*xs*,*ys*)) =
min(*length*(*xs*), *length*(*ys*))
⟨*proof*⟩
declare *length-zip* [*simp*]

lemma *zip-append1* [*rule-format*]:
⟦*ys*:*list*(*A*); *zs*:*list*(*B*)⟧ ⟹
∀ *xs* ∈ *list*(*A*). *zip*(*xs* @ *ys*, *zs*) =
zip(*xs*, *take*(*length*(*xs*), *zs*)) @ *zip*(*ys*, *drop*(*length*(*xs*),*zs*))
⟨*proof*⟩

lemma *zip-append2* [*rule-format*]:
⟦*xs*:*list*(*A*); *zs*:*list*(*B*)⟧ ⟹ ∀ *ys* ∈ *list*(*B*). *zip*(*xs*, *ys*@*zs*) =
zip(*take*(*length*(*ys*), *xs*), *ys*) @ *zip*(*drop*(*length*(*ys*), *xs*), *zs*)
⟨*proof*⟩

lemma *zip-append* [*simp*]:

$[\![length(xs) = length(us);\ length(ys) = length(vs);$
$xs{:}list(A);\ us{:}list(B);\ ys{:}list(A);\ vs{:}list(B)]\!]$
$\Longrightarrow zip(xs@ys,us@vs) = zip(xs,\ us)\ @\ zip(ys,\ vs)$
⟨*proof*⟩

lemma *zip-rev* [*rule-format*]:
$ys{:}list(B) \Longrightarrow \forall\, xs \in list(A).$
$length(xs) = length(ys) \longrightarrow zip(rev(xs),\ rev(ys)) = rev(zip(xs,\ ys))$
⟨*proof*⟩
declare *zip-rev* [*simp*]

lemma *nth-zip* [*rule-format*]:
$ys{:}list(B) \Longrightarrow \forall\, i \in nat.\ \forall\, xs \in list(A).$
$i < length(xs) \longrightarrow i < length(ys) \longrightarrow$
$nth(i,zip(xs,\ ys)) = <nth(i,xs),nth(i,\ ys)>$
⟨*proof*⟩
declare *nth-zip* [*simp*]

lemma *set-of-list-zip* [*rule-format*]:
$[\![xs{:}list(A);\ ys{:}list(B);\ i \in nat]\!]$
$\Longrightarrow set\text{-}of\text{-}list(zip(xs,\ ys)) =$
$\{\langle x,\ y\rangle{:}A{*}B.\ \exists\, i{\in}nat.\ i < min(length(xs),\ length(ys))$
$\wedge\ x = nth(i,\ xs) \wedge\ y = nth(i,\ ys)\}$
⟨*proof*⟩

lemma *list-update-Nil* [*simp*]: $i \in nat \Longrightarrow list\text{-}update(Nil,\ i,\ v) = Nil$
⟨*proof*⟩

lemma *list-update-Cons-0* [*simp*]: $list\text{-}update(Cons(x,\ xs),\ 0,\ v) = Cons(v,\ xs)$
⟨*proof*⟩

lemma *list-update-Cons-succ* [*simp*]:
$n \in nat \Longrightarrow$
$list\text{-}update(Cons(x,\ xs),\ succ(n),\ v) = Cons(x,\ list\text{-}update(xs,\ n,\ v))$
⟨*proof*⟩

lemma *list-update-type* [*rule-format*]:
$[\![xs{:}list(A);\ v \in A]\!] \Longrightarrow \forall\, n \in nat.\ list\text{-}update(xs,\ n,\ v){:}list(A)$
⟨*proof*⟩
declare *list-update-type* [*simp*,*TC*]

lemma *length-list-update* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, i \in nat.\ length(list\text{-}update(xs,\ i,\ v)) = length(xs)$
⟨*proof*⟩
declare *length-list-update* [*simp*]

lemma *nth-list-update* [*rule-format*]:
$[\![xs{:}list(A)]\!] \Longrightarrow \forall\, i \in nat.\ \forall\, j \in nat.\ i < length(xs) \longrightarrow$
$nth(j,\ list\text{-}update(xs,\ i,\ x)) = (if\ i{=}j\ then\ x\ else\ nth(j,\ xs))$
⟨*proof*⟩

lemma *nth-list-update-eq* [*simp*]:
$[\![i < length(xs);\ xs{:}list(A)]\!] \Longrightarrow nth(i,\ list\text{-}update(xs,\ i{,}x)) = x$
⟨*proof*⟩

lemma *nth-list-update-neq* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, i \in nat.\ \forall\, j \in nat.\ i \neq j \longrightarrow nth(j,\ list\text{-}update(xs{,}i{,}x)) = nth(j{,}xs)$
⟨*proof*⟩
declare *nth-list-update-neq* [*simp*]

lemma *list-update-overwrite* [*rule-format*]:
$xs{:}list(A) \Longrightarrow \forall\, i \in nat.\ i < length(xs)$
$\longrightarrow list\text{-}update(list\text{-}update(xs,\ i,\ x),\ i,\ y) = list\text{-}update(xs,\ i{,}y)$
⟨*proof*⟩
declare *list-update-overwrite* [*simp*]

lemma *list-update-same-conv* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, i \in nat.\ i < length(xs) \longrightarrow$
$(list\text{-}update(xs,\ i,\ x) = xs) \longleftrightarrow (nth(i,\ xs) = x)$
⟨*proof*⟩

lemma *update-zip* [*rule-format*]:
$ys{:}list(B) \Longrightarrow$
$\forall\, i \in nat.\ \forall\, xy \in A{*}B.\ \forall\, xs \in list(A).$
$length(xs) = length(ys) \longrightarrow$
$list\text{-}update(zip(xs,\ ys),\ i,\ xy) = zip(list\text{-}update(xs,\ i,\ fst(xy)),$
$list\text{-}update(ys,\ i,\ snd(xy)))$
⟨*proof*⟩

lemma *set-update-subset-cons* [*rule-format*]:
$xs{:}list(A) \Longrightarrow$
$\forall\, i \in nat.\ set\text{-}of\text{-}list(list\text{-}update(xs,\ i,\ x)) \subseteq cons(x,\ set\text{-}of\text{-}list(xs))$
⟨*proof*⟩

lemma *set-of-list-update-subsetI*:
$[\![set\text{-}of\text{-}list(xs) \subseteq A;\ xs{:}list(A);\ x \in A;\ i \in nat]\!]$
$\Longrightarrow set\text{-}of\text{-}list(list\text{-}update(xs,\ i{,}x)) \subseteq A$
⟨*proof*⟩

lemma *upt-rec*:

$j \in nat \Longrightarrow upt(i,j) = (if\ i<j\ then\ Cons(i,\ upt(succ(i),\ j))\ else\ Nil)$
⟨*proof*⟩

lemma *upt-conv-Nil* [*simp*]: $[\![j \leq i;\ j \in nat]\!] \Longrightarrow upt(i,j) = Nil$
⟨*proof*⟩

lemma *upt-succ-append*:
$[\![i \leq j;\ j \in nat]\!] \Longrightarrow upt(i,succ(j)) = upt(i,\ j)@[j]$
⟨*proof*⟩

lemma *upt-conv-Cons*:
$[\![i<j;\ j \in nat]\!] \Longrightarrow upt(i,j) = Cons(i,upt(succ(i),j))$
⟨*proof*⟩

lemma *upt-type* [*simp*,*TC*]: $j \in nat \Longrightarrow upt(i,j):list(nat)$
⟨*proof*⟩

lemma *upt-add-eq-append*:
$[\![i \leq j;\ j \in nat;\ k \in nat]\!] \Longrightarrow upt(i,\ j\ \#+k) = upt(i,j)@upt(j,j\#+k)$
⟨*proof*⟩

lemma *length-upt* [*simp*]: $[\![i \in nat;\ j \in nat]\!] \Longrightarrow length(upt(i,j)) = j\ \#-\ i$
⟨*proof*⟩

lemma *nth-upt* [*simp*]:
$[\![i \in nat;\ j \in nat;\ k \in nat;\ i\ \#+\ k < j]\!] \Longrightarrow nth(k,\ upt(i,j)) = i\ \#+\ k$
⟨*proof*⟩

lemma *take-upt* [*rule-format*]:
$[\![m \in nat;\ n \in nat]\!] \Longrightarrow$
$\forall\, i \in nat.\ i\ \#+\ m \leq n \longrightarrow take(m,\ upt(i,n)) = upt(i,i\#+m)$
⟨*proof*⟩
declare *take-upt* [*simp*]

lemma *map-succ-upt*:
$[\![m \in nat;\ n \in nat]\!] \Longrightarrow map(succ,\ upt(m,n)) = upt(succ(m),\ succ(n))$
⟨*proof*⟩

lemma *nth-map* [*rule-format*]:
$xs:list(A) \Longrightarrow$
$\forall\, n \in nat.\ n < length(xs) \longrightarrow nth(n,\ map(f,\ xs)) = f(nth(n,\ xs))$
⟨*proof*⟩
declare *nth-map* [*simp*]

lemma *nth-map-upt* [*rule-format*]:
$[\![m \in nat;\ n \in nat]\!] \Longrightarrow$
$\forall\, i \in nat.\ i < n\ \#-\ m \longrightarrow nth(i,\ map(f,\ upt(m,n))) = f(m\ \#+\ i)$

⟨*proof*⟩

definition
sublist :: [*i*, *i*] ⇒ *i* **where**
sublist(*xs*, *A*)≡
map(*fst*, (*filter*(λ*p*. *snd*(*p*): *A*, *zip*(*xs*, *upt*(0,*length*(*xs*))))))

lemma *sublist-0* [*simp*]: *xs*:*list*(*A*) ⟹*sublist*(*xs*, 0) =*Nil*
⟨*proof*⟩

lemma *sublist-Nil* [*simp*]: *sublist*(*Nil*, *A*) = *Nil*
⟨*proof*⟩

lemma *sublist-shift-lemma*:
⟦*xs*:*list*(*B*); *i* ∈ *nat*⟧ ⟹
map(*fst*, *filter*(λ*p*. *snd*(*p*):*A*, *zip*(*xs*, *upt*(*i*,*i* #+ *length*(*xs*))))) =
map(*fst*, *filter*(λ*p*. *snd*(*p*):*nat* ∧ *snd*(*p*) #+ *i* ∈ *A*, *zip*(*xs*,*upt*(0,*length*(*xs*)))))
⟨*proof*⟩

lemma *sublist-type* [*simp*,*TC*]:
xs:*list*(*B*) ⟹ *sublist*(*xs*, *A*):*list*(*B*)
⟨*proof*⟩

lemma *upt-add-eq-append2*:
⟦*i* ∈ *nat*; *j* ∈ *nat*⟧ ⟹ *upt*(0, *i* #+ *j*) = *upt*(0, *i*) @ *upt*(*i*, *i* #+ *j*)
⟨*proof*⟩

lemma *sublist-append*:
⟦*xs*:*list*(*B*); *ys*:*list*(*B*)⟧ ⟹
sublist(*xs*@*ys*, *A*) = *sublist*(*xs*, *A*) @ *sublist*(*ys*, {*j* ∈ *nat*. *j* #+ *length*(*xs*): *A*})
⟨*proof*⟩

lemma *sublist-Cons*:
⟦*xs*:*list*(*B*); *x* ∈ *B*⟧ ⟹
sublist(*Cons*(*x*, *xs*), *A*) =
(*if* 0 ∈ *A* *then* [*x*] *else* []) @ *sublist*(*xs*, {*j* ∈ *nat*. *succ*(*j*) ∈ *A*})
⟨*proof*⟩

lemma *sublist-singleton* [*simp*]:
sublist([*x*], *A*) = (*if* 0 ∈ *A* *then* [*x*] *else* [])
⟨*proof*⟩

lemma *sublist-upt-eq-take* [*rule-format*]:
xs:*list*(*A*) ⟹ ∀ *n*∈*nat*. *sublist*(*xs*,*n*) = *take*(*n*,*xs*)
⟨*proof*⟩
declare *sublist-upt-eq-take* [*simp*]

lemma *sublist-Int-eq*:
  $xs \in list(B) \Longrightarrow sublist(xs,\ A \cap nat) = sublist(xs,\ A)$
⟨*proof*⟩

Repetition of a List Element

consts *repeat* :: $[i,i]\Rightarrow i$
primrec
  $repeat(a,0) = []$

  $repeat(a,succ(n)) = Cons(a,repeat(a,n))$

lemma *length-repeat*: $n \in nat \Longrightarrow length(repeat(a,n)) = n$
⟨*proof*⟩

lemma *repeat-succ-app*: $n \in nat \Longrightarrow repeat(a,succ(n)) = repeat(a,n)\ @\ [a]$
⟨*proof*⟩

lemma *repeat-type* [*TC*]: $[\![a \in A;\ n \in nat]\!] \Longrightarrow repeat(a,n) \in list(A)$
⟨*proof*⟩

end

30 Equivalence Relations

theory *EquivClass* **imports** *Trancl Perm* **begin**

definition
  quotient :: $[i,i]\Rightarrow i$ (**infixl** ‹'//'› *90*) **where**
    $A//r \equiv \{r``\{x\}\ .\ x \in A\}$

definition
  congruent :: $[i,i\Rightarrow i]\Rightarrow o$ **where**
    $congruent(r,b) \equiv \forall\, y\ z.\ \langle y,z\rangle{:}r \longrightarrow b(y){=}b(z)$

definition
  congruent2 :: $[i,i,[i,i]\Rightarrow i]\Rightarrow o$ **where**
    $congruent2(r1,r2,b) \equiv \forall\, y1\ z1\ y2\ z2.$
      $\langle y1,z1\rangle{:}r1 \longrightarrow \langle y2,z2\rangle{:}r2 \longrightarrow b(y1,y2) = b(z1,z2)$

abbreviation
  RESPECTS ::$[i\Rightarrow i,\ i] \Rightarrow o$ (**infixr** ‹*respects*› *80*) **where**
  $f\ respects\ r \equiv congruent(r,f)$

abbreviation
  RESPECTS2 ::$[i\Rightarrow i\Rightarrow i,\ i] \Rightarrow o$ (**infixr** ‹*respects2* › *80*) **where**
  $f\ respects2\ r \equiv congruent2(r,r,f)$
  — Abbreviation for the common case where the relations are identical

30.1 Suppes, Theorem 70: r is an equiv relation iff $converse(r)$ O $r = r$

lemma *sym-trans-comp-subset*:
$[\![sym(r);\ trans(r)]\!] \Longrightarrow converse(r)\ O\ r \subseteq r$
⟨*proof*⟩

lemma *refl-comp-subset*:
$[\![refl(A,r);\ r \subseteq A*A]\!] \Longrightarrow r \subseteq converse(r)\ O\ r$
⟨*proof*⟩

lemma *equiv-comp-eq*:
$equiv(A,r) \Longrightarrow converse(r)\ O\ r = r$
⟨*proof*⟩

lemma *comp-equivI*:
$[\![converse(r)\ O\ r = r;\ \ domain(r) = A]\!] \Longrightarrow equiv(A,r)$
⟨*proof*⟩

lemma *equiv-class-subset*:
$[\![sym(r);\ \ trans(r);\ \ \langle a,b\rangle : r]\!] \Longrightarrow r``\{a\} \subseteq r``\{b\}$
⟨*proof*⟩

lemma *equiv-class-eq*:
$[\![equiv(A,r);\ \ \langle a,b\rangle : r]\!] \Longrightarrow r``\{a\} = r``\{b\}$
⟨*proof*⟩

lemma *equiv-class-self*:
$[\![equiv(A,r);\ \ a \in A]\!] \Longrightarrow a \in r``\{a\}$
⟨*proof*⟩

lemma *subset-equiv-class*:
$[\![equiv(A,r);\ \ r``\{b\} \subseteq r``\{a\};\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle : r$
⟨*proof*⟩

lemma *eq-equiv-class*: $[\![r``\{a\} = r``\{b\};\ \ equiv(A,r);\ \ b \in A]\!] \Longrightarrow \langle a,b\rangle : r$
⟨*proof*⟩

lemma *equiv-class-nondisjoint*:
$[\![equiv(A,r);\ \ x : (r``\{a\} \cap r``\{b\})]\!] \Longrightarrow \langle a,b\rangle : r$
⟨*proof*⟩

lemma *equiv-type*: $equiv(A,r) \Longrightarrow r \subseteq A*A$
⟨*proof*⟩

lemma *equiv-class-eq-iff*:
$equiv(A,r) \Longrightarrow \langle x,y\rangle\colon r \longleftrightarrow r``\{x\} = r``\{y\} \wedge x \in A \wedge y \in A$
⟨*proof*⟩

lemma *eq-equiv-class-iff*:
$[\![equiv(A,r);\ x \in A;\ y \in A]\!] \Longrightarrow r``\{x\} = r``\{y\} \longleftrightarrow \langle x,y\rangle\colon r$
⟨*proof*⟩

lemma *quotientI* [*TC*]: $x \in A \Longrightarrow r``\{x\}\colon A//r$
⟨*proof*⟩

lemma *quotientE*:
$[\![X \in A//r;\ \bigwedge x.\ [\![X = r``\{x\};\ x \in A]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *Union-quotient*:
$equiv(A,r) \Longrightarrow \bigcup(A//r) = A$
⟨*proof*⟩

lemma *quotient-disj*:
$[\![equiv(A,r);\ X \in A//r;\ Y \in A//r]\!] \Longrightarrow X=Y \mid (X \cap Y \subseteq 0)$
⟨*proof*⟩

30.2 Defining Unary Operations upon Equivalence Classes

lemma *UN-equiv-class*:
$[\![equiv(A,r);\ b\ respects\ r;\ a \in A]\!] \Longrightarrow (\bigcup x{\in}r``\{a\}.\ b(x)) = b(a)$
⟨*proof*⟩

lemma *UN-equiv-class-type*:
$[\![equiv(A,r);\ b\ respects\ r;\ X \in A//r;\ \bigwedge x.\ x \in A \Longrightarrow b(x) \in B]\!]$
$\Longrightarrow (\bigcup x{\in}X.\ b(x)) \in B$
⟨*proof*⟩

lemma *UN-equiv-class-inject*:
$[\![equiv(A,r);\ \ b\ respects\ r;$
$(\bigcup x{\in}X.\ b(x))=(\bigcup y{\in}Y.\ b(y));\ X \in A//r;\ Y \in A//r;$
$\bigwedge x\ y.\ [\![x \in A;\ y \in A;\ b(x)=b(y)]\!] \Longrightarrow \langle x,y\rangle{:}r]\!]$
$\Longrightarrow X=Y$
⟨*proof*⟩

30.3 Defining Binary Operations upon Equivalence Classes

lemma *congruent2-implies-congruent*:
$[\![equiv(A,r1);\ \ congruent2(r1,r2,b);\ \ a \in A]\!] \Longrightarrow congruent(r2,b(a))$
⟨*proof*⟩

lemma *congruent2-implies-congruent-UN*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);\ \ congruent2(r1,r2,b);\ \ a \in A2]\!] \Longrightarrow$
$congruent(r1,\ \lambda x1.\ \bigcup x2 \in r2``\{a\}.\ b(x1,x2))$
⟨*proof*⟩

lemma *UN-equiv-class2*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);\ \ congruent2(r1,r2,b);\ \ a1 : A1;\ \ a2 : A2]\!]$
$\Longrightarrow (\bigcup x1 \in r1``\{a1\}.\ \bigcup x2 \in r2``\{a2\}.\ b(x1,x2)) = b(a1,a2)$
⟨*proof*⟩

lemma *UN-equiv-class-type2*:
$[\![equiv(A,r);\ \ b\ respects2\ r;$
$X1 : A//r;\ \ X2 : A//r;$
$\bigwedge x1\ x2.\ [\![x1 : A;\ x2 : A]\!] \Longrightarrow b(x1,x2) \in B$
$]\!] \Longrightarrow (\bigcup x1 \in X1.\ \bigcup x2 \in X2.\ b(x1,x2)) \in B$
⟨*proof*⟩

lemma *congruent2I*:
$[\![equiv(A1,r1);\ \ equiv(A2,r2);$
$\bigwedge y\ z\ w.\ [\![w \in A2;\ \ \langle y,z\rangle \in r1]\!] \Longrightarrow b(y,w) = b(z,w);$
$\bigwedge y\ z\ w.\ [\![w \in A1;\ \ \langle y,z\rangle \in r2]\!] \Longrightarrow b(w,y) = b(w,z)$
$]\!] \Longrightarrow congruent2(r1,r2,b)$
⟨*proof*⟩

lemma *congruent2-commuteI*:
assumes *equivA*: $equiv(A,r)$
and *commute*: $\bigwedge y\ z.\ [\![y \in A;\ \ z \in A]\!] \Longrightarrow b(y,z) = b(z,y)$
and *congt*: $\bigwedge y\ z\ w.\ [\![w \in A;\ \ \langle y,z\rangle : r]\!] \Longrightarrow b(w,y) = b(w,z)$
shows *b respects2 r*
⟨*proof*⟩

lemma *congruent-commuteI*:
$[\![equiv(A,r);\ \ Z \in A//r;$
$\bigwedge w.\ [\![w \in A]\!] \Longrightarrow congruent(r,\ \lambda z.\ b(w,z));$
$\bigwedge x\ y.\ [\![x \in A;\ \ y \in A]\!] \Longrightarrow b(y,x) = b(x,y)$
$]\!] \Longrightarrow congruent(r,\ \lambda w.\ \bigcup z \in Z.\ b(w,z))$
⟨*proof*⟩

end

31 The Integers as Equivalence Classes Over Pairs of Natural Numbers

theory *Int* **imports** *EquivClass ArithSimp* **begin**

definition
intrel :: *i* **where**
intrel ≡ {*p* ∈ (*nat*∗*nat*)∗(*nat*∗*nat*).
∃ *x1 y1 x2 y2*. *p*=<⟨*x1*,*y1*⟩,⟨*x2*,*y2*⟩> ∧ *x1*#+*y2* = *x2*#+*y1*}

definition
int :: *i* **where**
int ≡ (*nat*∗*nat*)//*intrel*

definition
int-of :: *i*⇒*i* — coercion from nat to int (‹$# -› [*80*] *80*) **where**
$# *m* ≡ *intrel* `` {<*natify*(*m*), *0*>}

definition
intify :: *i*⇒*i* — coercion from ANYTHING to int **where**
intify(*m*) ≡ *if m* ∈ *int then m else* $#*0*

definition
raw-zminus :: *i*⇒*i* **where**
raw-zminus(*z*) ≡ ⋃⟨*x*,*y*⟩∈*z*. *intrel*``{⟨*y*,*x*⟩}

definition
zminus :: *i*⇒*i* (‹$− -› [*80*] *80*) **where**
$− *z* ≡ *raw-zminus* (*intify*(*z*))

definition
znegative :: *i*⇒*o* **where**
znegative(*z*) ≡ ∃ *x y*. *x*<*y* ∧ *y*∈*nat* ∧ ⟨*x*,*y*⟩∈*z*

definition
iszero :: *i*⇒*o* **where**
iszero(*z*) ≡ *z* = $# *0*

definition
raw-nat-of :: *i*⇒*i* **where**
raw-nat-of(*z*) ≡ *natify* (⋃⟨*x*,*y*⟩∈*z*. *x*#−*y*)

definition
nat-of :: *i*⇒*i* **where**
nat-of(*z*) ≡ *raw-nat-of* (*intify*(*z*))

definition
zmagnitude :: *i*⇒*i* **where**
— could be replaced by an absolute value function from int to int?

zmagnitude(*z*) ≡
THE m. m∈*nat* ∧ ((¬ *znegative*(*z*) ∧ *z* = $# *m*) |
(*znegative*(*z*) ∧ $− *z* = $# *m*))

definition
raw-zmult :: [*i*,*i*]⇒*i* **where**

raw-zmult(*z1*,*z2*) ≡
⋃*p1*∈*z1*. ⋃*p2*∈*z2*. *split*(λ*x1 y1*. *split*(λ*x2 y2*.
intrel``{<*x1*#∗*x2* #+ *y1*#∗*y2*, *x1*#∗*y2* #+ *y1*#∗*x2*>}, *p2*), *p1*)

definition
zmult :: [*i*,*i*]⇒*i* (**infixl** ‹$∗› 70) **where**
z1 $∗ *z2* ≡ *raw-zmult* (*intify*(*z1*),*intify*(*z2*))

definition
raw-zadd :: [*i*,*i*]⇒*i* **where**
raw-zadd (*z1*, *z2*) ≡
⋃*z1*∈*z1*. ⋃*z2*∈*z2*. *let* ⟨*x1*,*y1*⟩=*z1*; ⟨*x2*,*y2*⟩=*z2*
in intrel``{<*x1*#+*x2*, *y1*#+*y2*>}

definition
zadd :: [*i*,*i*]⇒*i* (**infixl** ‹$+› 65) **where**
z1 $+ *z2* ≡ *raw-zadd* (*intify*(*z1*),*intify*(*z2*))

definition
zdiff :: [*i*,*i*]⇒*i* (**infixl** ‹$−› 65) **where**
z1 $− *z2* ≡ *z1* $+ *zminus*(*z2*)

definition
zless :: [*i*,*i*]⇒*o* (**infixl** ‹$<› 50) **where**
z1 $< *z2* ≡ *znegative*(*z1* $− *z2*)

definition
zle :: [*i*,*i*]⇒*o* (**infixl** ‹$≤› 50) **where**
z1 $≤ *z2* ≡ *z1* $< *z2* | *intify*(*z1*)=*intify*(*z2*)

declare *quotientE* [*elim*!]

31.1 Proving that *intrel* is an equivalence relation

lemma *intrel-iff* [*simp*]:
<⟨*x1*,*y1*⟩,⟨*x2*,*y2*⟩>: *intrel* ⟷
x1∈*nat* ∧ *y1*∈*nat* ∧ *x2*∈*nat* ∧ *y2*∈*nat* ∧ *x1*#+*y2* = *x2*#+*y1*
⟨*proof*⟩

lemma *intrelI* [*intro*!]:
⟦*x1*#+*y2* = *x2*#+*y1*; *x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧

$\Longrightarrow$ $<\langle x1,y1\rangle,\langle x2,y2\rangle>$: *intrel*
⟨*proof*⟩

lemma *intrelE* [*elim*!]:
$[\![p \in intrel;$
$\bigwedge x1\ y1\ x2\ y2.\ [\![p = <\langle x1,y1\rangle,\langle x2,y2\rangle>;\ x1\#+y2 = x2\#+y1;$
$x1 \in nat;\ y1 \in nat;\ x2 \in nat;\ y2 \in nat]\!] \Longrightarrow Q]\!]$
$\Longrightarrow Q$
⟨*proof*⟩

lemma *int-trans-lemma*:
$[\![x1\ \#+\ y2 = x2\ \#+\ y1;\ x2\ \#+\ y3 = x3\ \#+\ y2]\!] \Longrightarrow x1\ \#+\ y3 = x3\ \#+\ y1$
⟨*proof*⟩

lemma *equiv-intrel*: *equiv*(*nat*∗*nat*, *intrel*)
⟨*proof*⟩

lemma *image-intrel-int*: $[\![m \in nat;\ n \in nat]\!] \Longrightarrow intrel\ ``\ \{\langle m,n\rangle\} \in int$
⟨*proof*⟩

declare *equiv-intrel* [*THEN eq-equiv-class-iff*, *simp*]
declare *conj-cong* [*cong*]

lemmas *eq-intrelD* = *eq-equiv-class* [*OF - equiv-intrel*]

lemma *int-of-type* [*simp*,*TC*]: $\$\#m \in int$
⟨*proof*⟩

lemma *int-of-eq* [*iff*]: $(\$\#\ m = \$\#\ n) \longleftrightarrow natify(m)=natify(n)$
⟨*proof*⟩

lemma *int-of-inject*: $[\![\$\#m = \$\#n;\ \ m \in nat;\ \ n \in nat]\!] \Longrightarrow m=n$
⟨*proof*⟩

lemma *intify-in-int* [*iff*,*TC*]: $intify(x) \in int$
⟨*proof*⟩

lemma *intify-ident* [*simp*]: $n \in int \Longrightarrow intify(n) = n$
⟨*proof*⟩

31.2 Collapsing rules: to remove *intify* from arithmetic expressions

lemma *intify-idem* [*simp*]: $intify(intify(x)) = intify(x)$
⟨*proof*⟩

lemma *int-of-natify* [*simp*]: $\$\# (natify(m)) = \$\# m$
⟨*proof*⟩

lemma *zminus-intify* [*simp*]: $\$- (intify(m)) = \$- m$
⟨*proof*⟩

lemma *zadd-intify1* [*simp*]: $intify(x) \$+ y = x \$+ y$
⟨*proof*⟩

lemma *zadd-intify2* [*simp*]: $x \$+ intify(y) = x \$+ y$
⟨*proof*⟩

lemma *zdiff-intify1* [*simp*]:$intify(x) \$- y = x \$- y$
⟨*proof*⟩

lemma *zdiff-intify2* [*simp*]:$x \$- intify(y) = x \$- y$
⟨*proof*⟩

lemma *zmult-intify1* [*simp*]:$intify(x) \$* y = x \$* y$
⟨*proof*⟩

lemma *zmult-intify2* [*simp*]:$x \$* intify(y) = x \$* y$
⟨*proof*⟩

lemma *zless-intify1* [*simp*]:$intify(x) \$< y \longleftrightarrow x \$< y$
⟨*proof*⟩

lemma *zless-intify2* [*simp*]:$x \$< intify(y) \longleftrightarrow x \$< y$
⟨*proof*⟩

lemma *zle-intify1* [*simp*]:$intify(x) \$\leq y \longleftrightarrow x \$\leq y$
⟨*proof*⟩

lemma *zle-intify2* [*simp*]:$x \$\leq intify(y) \longleftrightarrow x \$\leq y$
⟨*proof*⟩

31.3 *zminus*: unary negation on *int*

lemma *zminus-congruent*: $(\lambda\langle x,y\rangle.\ intrel``\{\langle y,x\rangle\})$ *respects intrel*
⟨*proof*⟩

lemma *raw-zminus-type*: $z \in int \Longrightarrow raw\text{-}zminus(z) \in int$
⟨*proof*⟩

lemma *zminus-type* [*TC*,*iff*]: $\$-z \in int$
⟨*proof*⟩

lemma *raw-zminus-inject*:
$[\![raw\text{-}zminus(z) = raw\text{-}zminus(w);\ \ z \in int;\ \ w \in int]\!] \Longrightarrow z=w$
⟨*proof*⟩

lemma *zminus-inject-intify* [*dest*!]: $\$-z = \$-w \Longrightarrow intify(z) = intify(w)$
⟨*proof*⟩

lemma *zminus-inject*: $[\![\$-z = \$-w;\ \ z \in int;\ \ w \in int]\!] \Longrightarrow z=w$
⟨*proof*⟩

lemma *raw-zminus*:
$[\![x\in nat;\ \ y\in nat]\!] \Longrightarrow raw\text{-}zminus(intrel``\{\langle x,y\rangle\}) = intrel\ ``\ \{\langle y,x\rangle\}$
⟨*proof*⟩

lemma *zminus*:
$[\![x\in nat;\ \ y\in nat]\!]$
$\Longrightarrow \$-\ (intrel``\{\langle x,y\rangle\}) = intrel\ ``\ \{\langle y,x\rangle\}$
⟨*proof*⟩

lemma *raw-zminus-zminus*: $z \in int \Longrightarrow raw\text{-}zminus\ (raw\text{-}zminus(z)) = z$
⟨*proof*⟩

lemma *zminus-zminus-intify* [*simp*]: $\$-\ (\$-\ z) = intify(z)$
⟨*proof*⟩

lemma *zminus-int0* [*simp*]: $\$-\ (\$\#0) = \$\#0$
⟨*proof*⟩

lemma *zminus-zminus*: $z \in int \Longrightarrow \$-\ (\$-\ z) = z$
⟨*proof*⟩

31.4 *znegative*: the test for negative integers

lemma *znegative*: $[\![x\in nat;\ y\in nat]\!] \Longrightarrow znegative(intrel``\{\langle x,y\rangle\}) \longleftrightarrow x<y$
⟨*proof*⟩

lemma *not-znegative-int-of* [*iff*]: $\neg\ znegative(\$\#\ n)$
⟨*proof*⟩

lemma *znegative-zminus-int-of* [*simp*]: *znegative*($− $# *succ*(*n*))
⟨*proof*⟩

lemma *not-znegative-imp-zero*: ¬ *znegative*($− $# *n*) ⟹ *natify*(*n*)=*0*
⟨*proof*⟩

31.5 *nat-of*: Coercion of an Integer to a Natural Number

lemma *nat-of-intify* [*simp*]: *nat-of*(*intify*(*z*)) = *nat-of*(*z*)
⟨*proof*⟩

lemma *nat-of-congruent*: (λ*x*. (λ⟨*x*,*y*⟩. *x* #− *y*)(*x*)) *respects intrel*
⟨*proof*⟩

lemma *raw-nat-of*:
⟦*x*∈*nat*; *y*∈*nat*⟧ ⟹ *raw-nat-of*(*intrel*``{⟨*x*,*y*⟩}) = *x*#−*y*
⟨*proof*⟩

lemma *raw-nat-of-int-of*: *raw-nat-of*($# *n*) = *natify*(*n*)
⟨*proof*⟩

lemma *nat-of-int-of* [*simp*]: *nat-of*($# *n*) = *natify*(*n*)
⟨*proof*⟩

lemma *raw-nat-of-type*: *raw-nat-of*(*z*) ∈ *nat*
⟨*proof*⟩

lemma *nat-of-type* [*iff*,*TC*]: *nat-of*(*z*) ∈ *nat*
⟨*proof*⟩

31.6 zmagnitude: magnitide of an integer, as a natural number

lemma *zmagnitude-int-of* [*simp*]: *zmagnitude*($# *n*) = *natify*(*n*)
⟨*proof*⟩

lemma *natify-int-of-eq*: *natify*(*x*)=*n* ⟹ $#*x* = $# *n*
⟨*proof*⟩

lemma *zmagnitude-zminus-int-of* [*simp*]: *zmagnitude*($− $# *n*) = *natify*(*n*)
⟨*proof*⟩

lemma *zmagnitude-type* [*iff*,*TC*]: *zmagnitude*(*z*)∈*nat*
⟨*proof*⟩

lemma *not-zneg-int-of*:
⟦*z* ∈ *int*; ¬ *znegative*(*z*)⟧ ⟹ ∃ *n*∈*nat*. *z* = $# *n*
⟨*proof*⟩

lemma *not-zneg-mag* [*simp*]:
$[\![z \in int;\ \neg\ znegative(z)]\!] \Longrightarrow \$\#\ (zmagnitude(z)) = z$
⟨*proof*⟩

lemma *zneg-int-of*:
$[\![znegative(z);\ z \in int]\!] \Longrightarrow \exists\, n{\in}nat.\ z = \$-\ (\$\#\ succ(n))$
⟨*proof*⟩

lemma *zneg-mag* [*simp*]:
$[\![znegative(z);\ z \in int]\!] \Longrightarrow \$\#\ (zmagnitude(z)) = \$-\ z$
⟨*proof*⟩

lemma *int-cases*: $z \in int \Longrightarrow \exists\, n{\in}nat.\ z = \$\#\ n \mid z = \$-\ (\$\#\ succ(n))$
⟨*proof*⟩

lemma *not-zneg-raw-nat-of*:
$[\![\neg\ znegative(z);\ z \in int]\!] \Longrightarrow \$\#\ (raw\text{-}nat\text{-}of(z)) = z$
⟨*proof*⟩

lemma *not-zneg-nat-of-intify*:
$\neg\ znegative(intify(z)) \Longrightarrow \$\#\ (nat\text{-}of(z)) = intify(z)$
⟨*proof*⟩

lemma *not-zneg-nat-of*: $[\![\neg\ znegative(z);\ z \in int]\!] \Longrightarrow \$\#\ (nat\text{-}of(z)) = z$
⟨*proof*⟩

lemma *zneg-nat-of* [*simp*]: $znegative(intify(z)) \Longrightarrow nat\text{-}of(z) = 0$
⟨*proof*⟩

31.7 ($+): addition on int

Congruence Property for Addition

lemma *zadd-congruent2*:
$(\lambda z1\ z2.\ let\ \langle x1,y1\rangle{=}z1;\ \langle x2,y2\rangle{=}z2$
$\qquad in\ intrel``\{<x1\#{+}x2,\ y1\#{+}y2>\})$
respects2 intrel
⟨*proof*⟩

lemma *raw-zadd-type*: $[\![z \in int;\ \ w \in int]\!] \Longrightarrow raw\text{-}zadd(z,w) \in int$
⟨*proof*⟩

lemma *zadd-type* [*iff*, *TC*]: $z\ \$+\ w \in int$
⟨*proof*⟩

lemma *raw-zadd*:
$[\![x1{\in}nat;\ y1{\in}nat;\ \ x2{\in}nat;\ y2{\in}nat]\!]$
$\Longrightarrow raw\text{-}zadd\ (intrel``\{\langle x1,y1\rangle\},\ intrel``\{\langle x2,y2\rangle\}) =$
$\quad intrel\ ``\ \{<x1\#{+}x2,\ y1\#{+}y2>\}$

⟨*proof*⟩

lemma *zadd*:
[⟦*x1*∈*nat*; *y1*∈*nat*; *x2*∈*nat*; *y2*∈*nat*⟧
⟹ (*intrel*``{⟨*x1*,*y1*⟩}) $+ (*intrel*``{⟨*x2*,*y2*⟩}) =
intrel `` {<*x1*#+*x2*, *y1*#+*y2*>}
⟨*proof*⟩

lemma *raw-zadd-int0*: *z* ∈ *int* ⟹ *raw-zadd* ($#*0*,*z*) = *z*
⟨*proof*⟩

lemma *zadd-int0-intify* [*simp*]: $#*0* $+ *z* = *intify*(*z*)
⟨*proof*⟩

lemma *zadd-int0*: *z* ∈ *int* ⟹ $#*0* $+ *z* = *z*
⟨*proof*⟩

lemma *raw-zminus-zadd-distrib*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ $− *raw-zadd*(*z*,*w*) = *raw-zadd*($− *z*, $− *w*)
⟨*proof*⟩

lemma *zminus-zadd-distrib* [*simp*]: $− (*z* $+ *w*) = $− *z* $+ $− *w*
⟨*proof*⟩

lemma *raw-zadd-commute*:
⟦*z* ∈ *int*; *w* ∈ *int*⟧ ⟹ *raw-zadd*(*z*,*w*) = *raw-zadd*(*w*,*z*)
⟨*proof*⟩

lemma *zadd-commute*: *z* $+ *w* = *w* $+ *z*
⟨*proof*⟩

lemma *raw-zadd-assoc*:
⟦*z1*: *int*; *z2*: *int*; *z3*: *int*⟧
⟹ *raw-zadd* (*raw-zadd*(*z1*,*z2*),*z3*) = *raw-zadd*(*z1*,*raw-zadd*(*z2*,*z3*))
⟨*proof*⟩

lemma *zadd-assoc*: (*z1* $+ *z2*) $+ *z3* = *z1* $+ (*z2* $+ *z3*)
⟨*proof*⟩

lemma *zadd-left-commute*: *z1*$+(*z2*$+*z3*) = *z2*$+(*z1*$+*z3*)
⟨*proof*⟩

lemmas *zadd-ac* = *zadd-assoc zadd-commute zadd-left-commute*

lemma *int-of-add*: $# (*m* #+ *n*) = ($#*m*) $+ ($#*n*)
⟨*proof*⟩

lemma *int-succ-int-1*: $\$\#\ succ(m) = \$\#\ 1\ \$+\ (\$\#\ m)$
⟨*proof*⟩

lemma *int-of-diff*:
$[\![m{\in}nat;\ n \leq m]\!] \Longrightarrow \$\#\ (m\ \#-\ n) = (\$\#m)\ \$-\ (\$\#n)$
⟨*proof*⟩

lemma *raw-zadd-zminus-inverse*: $z \in int \Longrightarrow raw\text{-}zadd\ (z, \$-\ z) = \$\#0$
⟨*proof*⟩

lemma *zadd-zminus-inverse* [*simp*]: $z\ \$+\ (\$-\ z) = \$\#0$
⟨*proof*⟩

lemma *zadd-zminus-inverse2* [*simp*]: $(\$-\ z)\ \$+\ z = \$\#0$
⟨*proof*⟩

lemma *zadd-int0-right-intify* [*simp*]: $z\ \$+\ \$\#0 = intify(z)$
⟨*proof*⟩

lemma *zadd-int0-right*: $z \in int \Longrightarrow z\ \$+\ \$\#0 = z$
⟨*proof*⟩

31.8 ($\$*$): Integer Multiplication

Congruence property for multiplication

lemma *zmult-congruent2*:
$(\lambda p1\ p2.\ split(\lambda x1\ y1.\ split(\lambda x2\ y2.$
$intrel``\{<x1\#*x2\ \#+\ y1\#*y2,\ x1\#*y2\ \#+\ y1\#*x2>\},\ p2),\ p1))$
respects2 intrel
⟨*proof*⟩

lemma *raw-zmult-type*: $[\![z \in int;\ \ w \in int]\!] \Longrightarrow raw\text{-}zmult(z,w) \in int$
⟨*proof*⟩

lemma *zmult-type* [*iff*, *TC*]: $z\ \$*\ w \in int$
⟨*proof*⟩

lemma *raw-zmult*:
$[\![x1{\in}nat;\ y1{\in}nat;\ \ x2{\in}nat;\ y2{\in}nat]\!]$
$\Longrightarrow raw\text{-}zmult(intrel``\{\langle x1,y1\rangle\},\ intrel``\{\langle x2,y2\rangle\}) =$
$intrel\ ``\ \{<x1\#*x2\ \#+\ y1\#*y2,\ x1\#*y2\ \#+\ y1\#*x2>\}$
⟨*proof*⟩

lemma *zmult*:
$[\![x1{\in}nat;\ y1{\in}nat;\ \ x2{\in}nat;\ y2{\in}nat]\!]$
$\Longrightarrow (intrel``\{\langle x1,y1\rangle\})\ \$*\ (intrel``\{\langle x2,y2\rangle\}) =$
$intrel\ ``\ \{<x1\#*x2\ \#+\ y1\#*y2,\ x1\#*y2\ \#+\ y1\#*x2>\}$
⟨*proof*⟩

lemma *raw-zmult-int0*: $z \in int \Longrightarrow raw\text{-}zmult\ (\$\#0,z) = \$\#0$
⟨*proof*⟩

lemma *zmult-int0* [*simp*]: $\$\#0\ \$*\ z = \$\#0$
⟨*proof*⟩

lemma *raw-zmult-int1*: $z \in int \Longrightarrow raw\text{-}zmult\ (\$\#1,z) = z$
⟨*proof*⟩

lemma *zmult-int1-intify* [*simp*]: $\$\#1\ \$*\ z = intify(z)$
⟨*proof*⟩

lemma *zmult-int1*: $z \in int \Longrightarrow \$\#1\ \$*\ z = z$
⟨*proof*⟩

lemma *raw-zmult-commute*:
$[\![z \in int;\ \ w \in int]\!] \Longrightarrow raw\text{-}zmult(z,w) = raw\text{-}zmult(w,z)$
⟨*proof*⟩

lemma *zmult-commute*: $z\ \$*\ w = w\ \$*\ z$
⟨*proof*⟩

lemma *raw-zmult-zminus*:
$[\![z \in int;\ \ w \in int]\!] \Longrightarrow raw\text{-}zmult(\$-\ z,\ w) = \$-\ raw\text{-}zmult(z,\ w)$
⟨*proof*⟩

lemma *zmult-zminus* [*simp*]: $(\$-\ z)\ \$*\ w = \$-\ (z\ \$*\ w)$
⟨*proof*⟩

lemma *zmult-zminus-right* [*simp*]: $w\ \$*\ (\$-\ z) = \$-\ (w\ \$*\ z)$
⟨*proof*⟩

lemma *raw-zmult-assoc*:
$[\![z1\text{: }int;\ \ z2\text{: }int;\ \ z3\text{: }int]\!]$
$\Longrightarrow raw\text{-}zmult\ (raw\text{-}zmult(z1,z2),z3) = raw\text{-}zmult(z1,raw\text{-}zmult(z2,z3))$
⟨*proof*⟩

lemma *zmult-assoc*: $(z1\ \$*\ z2)\ \$*\ z3 = z1\ \$*\ (z2\ \$*\ z3)$
⟨*proof*⟩

lemma *zmult-left-commute*: $z1\$*(z2\$*z3) = z2\$*(z1\$*z3)$
⟨*proof*⟩

lemmas *zmult-ac = zmult-assoc zmult-commute zmult-left-commute*

lemma *raw-zadd-zmult-distrib*:

$[\![z1\colon int;\ \ z2\colon int;\ \ w \in int]\!]$
$\Longrightarrow raw\text{-}zmult(raw\text{-}zadd(z1,z2),\ w) =$
$raw\text{-}zadd\ (raw\text{-}zmult(z1,w),\ raw\text{-}zmult(z2,w))$
⟨*proof*⟩

lemma *zadd-zmult-distrib*: $(z1\ \$+\ z2)\ \$*\ w = (z1\ \$*\ w)\ \$+\ (z2\ \$*\ w)$
⟨*proof*⟩

lemma *zadd-zmult-distrib2*: $w\ \$*\ (z1\ \$+\ z2) = (w\ \$*\ z1)\ \$+\ (w\ \$*\ z2)$
⟨*proof*⟩

lemmas *int-typechecks* =
int-of-type zminus-type zmagnitude-type zadd-type zmult-type

lemma *zdiff-type* [*iff*,*TC*]: $z\ \$-\ w \in int$
⟨*proof*⟩

lemma *zminus-zdiff-eq* [*simp*]: $\$-\ (z\ \$-\ y) = y\ \$-\ z$
⟨*proof*⟩

lemma *zdiff-zmult-distrib*: $(z1\ \$-\ z2)\ \$*\ w = (z1\ \$*\ w)\ \$-\ (z2\ \$*\ w)$
⟨*proof*⟩

lemma *zdiff-zmult-distrib2*: $w\ \$*\ (z1\ \$-\ z2) = (w\ \$*\ z1)\ \$-\ (w\ \$*\ z2)$
⟨*proof*⟩

lemma *zadd-zdiff-eq*: $x\ \$+\ (y\ \$-\ z) = (x\ \$+\ y)\ \$-\ z$
⟨*proof*⟩

lemma *zdiff-zadd-eq*: $(x\ \$-\ y)\ \$+\ z = (x\ \$+\ z)\ \$-\ y$
⟨*proof*⟩

31.9 The "Less Than" Relation

lemma *zless-linear-lemma*:
$[\![z \in int;\ w \in int]\!] \Longrightarrow z\$<w \mid z=w \mid w\$<z$
⟨*proof*⟩

lemma *zless-linear*: $z\$<w \mid intify(z)=intify(w) \mid w\$<z$
⟨*proof*⟩

lemma *zless-not-refl* [*iff*]: $\neg\ (z\$<z)$
⟨*proof*⟩

lemma *neq-iff-zless*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (x \neq y) \longleftrightarrow (x\ \$<\ y \mid y\ \$<\ x)$
⟨*proof*⟩

lemma *zless-imp-intify-neq*: $w \$< z \Longrightarrow intify(w) \neq intify(z)$
⟨*proof*⟩

lemma *zless-imp-succ-zadd-lemma*:
$[\![w \$< z;\ w \in int;\ z \in int]\!] \Longrightarrow (\exists n{\in}nat.\ z = w \$+ \$\#(succ(n)))$
⟨*proof*⟩

lemma *zless-imp-succ-zadd*:
$w \$< z \Longrightarrow (\exists n{\in}nat.\ w \$+ \$\#(succ(n)) = intify(z))$
⟨*proof*⟩

lemma *zless-succ-zadd-lemma*:
$w \in int \Longrightarrow w \$< w \$+ \$\#\ succ(n)$
⟨*proof*⟩

lemma *zless-succ-zadd*: $w \$< w \$+ \$\#\ succ(n)$
⟨*proof*⟩

lemma *zless-iff-succ-zadd*:
$w \$< z \longleftrightarrow (\exists n{\in}nat.\ w \$+ \$\#(succ(n)) = intify(z))$
⟨*proof*⟩

lemma *zless-int-of* [*simp*]: $[\![m{\in}nat;\ n{\in}nat]\!] \Longrightarrow (\$\#m \$< \$\#n) \longleftrightarrow (m{<}n)$
⟨*proof*⟩

lemma *zless-trans-lemma*:
$[\![x \$< y;\ y \$< z;\ x \in int;\ y \in int;\ z \in int]\!] \Longrightarrow x \$< z$
⟨*proof*⟩

lemma *zless-trans* [*trans*]: $[\![x \$< y;\ y \$< z]\!] \Longrightarrow x \$< z$
⟨*proof*⟩

lemma *zless-not-sym*: $z \$< w \Longrightarrow \neg\ (w \$< z)$
⟨*proof*⟩

lemmas *zless-asym* = *zless-not-sym* [*THEN swap*]

lemma *zless-imp-zle*: $z \$< w \Longrightarrow z \$\leq w$
⟨*proof*⟩

lemma *zle-linear*: $z \$\leq w \mid w \$\leq z$
⟨*proof*⟩

31.10 Less Than or Equals

lemma *zle-refl*: $z \$\leq z$

⟨*proof*⟩

lemma *zle-eq-refl*: $x{=}y \Longrightarrow x \ \$\leq \ y$
⟨*proof*⟩

lemma *zle-anti-sym-intify*: $[\![x \ \$\leq \ y;\ y \ \$\leq \ x]\!] \Longrightarrow \mathit{intify}(x) = \mathit{intify}(y)$
⟨*proof*⟩

lemma *zle-anti-sym*: $[\![x \ \$\leq \ y;\ y \ \$\leq \ x;\ x \in \mathit{int};\ y \in \mathit{int}]\!] \Longrightarrow x{=}y$
⟨*proof*⟩

lemma *zle-trans-lemma*:
$[\![x \in \mathit{int};\ y \in \mathit{int};\ z \in \mathit{int};\ x \ \$\leq \ y;\ y \ \$\leq \ z]\!] \Longrightarrow x \ \$\leq \ z$
⟨*proof*⟩

lemma *zle-trans* [*trans*]: $[\![x \ \$\leq \ y;\ y \ \$\leq \ z]\!] \Longrightarrow x \ \$\leq \ z$
⟨*proof*⟩

lemma *zle-zless-trans* [*trans*]: $[\![i \ \$\leq \ j;\ j \ \$< \ k]\!] \Longrightarrow i \ \$< \ k$
⟨*proof*⟩

lemma *zless-zle-trans* [*trans*]: $[\![i \ \$< \ j;\ j \ \$\leq \ k]\!] \Longrightarrow i \ \$< \ k$
⟨*proof*⟩

lemma *not-zless-iff-zle*: $\neg \ (z \ \$< \ w) \longleftrightarrow (w \ \$\leq \ z)$
⟨*proof*⟩

lemma *not-zle-iff-zless*: $\neg \ (z \ \$\leq \ w) \longleftrightarrow (w \ \$< \ z)$
⟨*proof*⟩

31.11 More subtraction laws (for *zcompare-rls*)

lemma *zdiff-zdiff-eq*: $(x \ \$- \ y) \ \$- \ z = x \ \$- \ (y \ \$+ \ z)$
⟨*proof*⟩

lemma *zdiff-zdiff-eq2*: $x \ \$- \ (y \ \$- \ z) = (x \ \$+ \ z) \ \$- \ y$
⟨*proof*⟩

lemma *zdiff-zless-iff*: $(x\${-}y \ \$< \ z) \longleftrightarrow (x \ \$< \ z \ \$+ \ y)$
⟨*proof*⟩

lemma *zless-zdiff-iff*: $(x \ \$< \ z\${-}y) \longleftrightarrow (x \ \$+ \ y \ \$< \ z)$
⟨*proof*⟩

lemma *zdiff-eq-iff*: $[\![x \in \mathit{int};\ z \in \mathit{int}]\!] \Longrightarrow (x\${-}y = z) \longleftrightarrow (x = z \ \$+ \ y)$
⟨*proof*⟩

lemma *eq-zdiff-iff*: $[\![x \in \mathit{int};\ z \in \mathit{int}]\!] \Longrightarrow (x = z\${-}y) \longleftrightarrow (x \ \$+ \ y = z)$
⟨*proof*⟩

lemma *zdiff-zle-iff-lemma*:
$[\![x \in int;\ z \in int]\!] \Longrightarrow (x\$-y\ \$\leq\ z) \longleftrightarrow (x\ \$\leq\ z\ \$+\ y)$
⟨*proof*⟩

lemma *zdiff-zle-iff*: $(x\$-y\ \$\leq\ z) \longleftrightarrow (x\ \$\leq\ z\ \$+\ y)$
⟨*proof*⟩

lemma *zle-zdiff-iff-lemma*:
$[\![x \in int;\ z \in int]\!] \Longrightarrow (x\ \$\leq\ z\$-y) \longleftrightarrow (x\ \$+\ y\ \$\leq\ z)$
⟨*proof*⟩

lemma *zle-zdiff-iff*: $(x\ \$\leq\ z\$-y) \longleftrightarrow (x\ \$+\ y\ \$\leq\ z)$
⟨*proof*⟩

This list of rewrites simplifies (in)equalities by bringing subtractions to the top and then moving negative terms to the other side. Use with *zadd-ac*

lemmas *zcompare-rls* =
zdiff-def [*symmetric*]
zadd-zdiff-eq zdiff-zadd-eq zdiff-zdiff-eq zdiff-zdiff-eq2
zdiff-zless-iff zless-zdiff-iff zdiff-zle-iff zle-zdiff-iff
zdiff-eq-iff eq-zdiff-iff

31.12 Monotonicity and Cancellation Results for Instantiation of the CancelNumerals Simprocs

lemma *zadd-left-cancel*:
$[\![w \in int;\ w'\text{: } int]\!] \Longrightarrow (z\ \$+\ w' = z\ \$+\ w) \longleftrightarrow (w' = w)$
⟨*proof*⟩

lemma *zadd-left-cancel-intify* [*simp*]:
$(z\ \$+\ w' = z\ \$+\ w) \longleftrightarrow intify(w') = intify(w)$
⟨*proof*⟩

lemma *zadd-right-cancel*:
$[\![w \in int;\ w'\text{: } int]\!] \Longrightarrow (w'\ \$+\ z = w\ \$+\ z) \longleftrightarrow (w' = w)$
⟨*proof*⟩

lemma *zadd-right-cancel-intify* [*simp*]:
$(w'\ \$+\ z = w\ \$+\ z) \longleftrightarrow intify(w') = intify(w)$
⟨*proof*⟩

lemma *zadd-right-cancel-zless* [*simp*]: $(w'\ \$+\ z\ \$<\ w\ \$+\ z) \longleftrightarrow (w'\ \$<\ w)$
⟨*proof*⟩

lemma *zadd-left-cancel-zless* [*simp*]: $(z\ \$+\ w'\ \$<\ z\ \$+\ w) \longleftrightarrow (w'\ \$<\ w)$
⟨*proof*⟩

lemma *zadd-right-cancel-zle* [*simp*]: $(w'\ \$+\ z\ \$\leq\ w\ \$+\ z) \longleftrightarrow w'\ \$\leq\ w$

⟨*proof*⟩

lemma *zadd-left-cancel-zle* [*simp*]: $(z \$+ w' \$\leq z \$+ w) \longleftrightarrow w' \$\leq w$
⟨*proof*⟩

lemmas *zadd-zless-mono1* = *zadd-right-cancel-zless* [*THEN iffD2*]

lemmas *zadd-zless-mono2* = *zadd-left-cancel-zless* [*THEN iffD2*]

lemmas *zadd-zle-mono1* = *zadd-right-cancel-zle* [*THEN iffD2*]

lemmas *zadd-zle-mono2* = *zadd-left-cancel-zle* [*THEN iffD2*]

lemma *zadd-zle-mono*: $[\![w' \$\leq w; z' \$\leq z]\!] \Longrightarrow w' \$+ z' \$\leq w \$+ z$
⟨*proof*⟩

lemma *zadd-zless-mono*: $[\![w' \$< w; z' \$\leq z]\!] \Longrightarrow w' \$+ z' \$< w \$+ z$
⟨*proof*⟩

31.13 Comparison laws

lemma *zminus-zless-zminus* [*simp*]: $(\$- x \$< \$- y) \longleftrightarrow (y \$< x)$
⟨*proof*⟩

lemma *zminus-zle-zminus* [*simp*]: $(\$- x \$\leq \$- y) \longleftrightarrow (y \$\leq x)$
⟨*proof*⟩

31.13.1 More inequality lemmas

lemma *equation-zminus*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (x = \$- y) \longleftrightarrow (y = \$- x)$
⟨*proof*⟩

lemma *zminus-equation*: $[\![x \in int;\ y \in int]\!] \Longrightarrow (\$- x = y) \longleftrightarrow (\$- y = x)$
⟨*proof*⟩

lemma *equation-zminus-intify*: $(intify(x) = \$- y) \longleftrightarrow (intify(y) = \$- x)$
⟨*proof*⟩

lemma *zminus-equation-intify*: $(\$- x = intify(y)) \longleftrightarrow (\$- y = intify(x))$
⟨*proof*⟩

31.13.2 The next several equations are permutative: watch out!

lemma *zless-zminus*: $(x \$< \$- y) \longleftrightarrow (y \$< \$- x)$
⟨*proof*⟩

lemma *zminus-zless*: ($− $x $< $y) ⟷ ($− $y $< $x)
⟨*proof*⟩

lemma *zle-zminus*: ($x $≤ $− $y) ⟷ ($y $≤ $− $x)
⟨*proof*⟩

lemma *zminus-zle*: ($− $x $≤ $y) ⟷ ($− $y $≤ $x)
⟨*proof*⟩

end

32 Arithmetic on Binary Integers

theory *Bin*
imports *Int Datatype*
begin

consts *bin* :: *i*
datatype
bin = *Pls*
| *Min*
| *Bit* (*w* ∈ *bin*, *b* ∈ *bool*) (**infixl** ‹*BIT*› *90*)

consts
integ-of :: *i*⇒*i*
NCons :: [*i*,*i*]⇒*i*
bin-succ :: *i*⇒*i*
bin-pred :: *i*⇒*i*
bin-minus :: *i*⇒*i*
bin-adder :: *i*⇒*i*
bin-mult :: [*i*,*i*]⇒*i*

primrec
integ-of-Pls: *integ-of* (*Pls*) = $# *0*
integ-of-Min: *integ-of* (*Min*) = $−($#*1*)
integ-of-BIT: *integ-of* (*w BIT b*) = $#*b* $+ *integ-of*(*w*) $+ *integ-of*(*w*)

primrec
NCons-Pls: *NCons* (*Pls*,*b*) = *cond*(*b*,*Pls BIT b*,*Pls*)
NCons-Min: *NCons* (*Min*,*b*) = *cond*(*b*,*Min*,*Min BIT b*)
NCons-BIT: *NCons* (*w BIT c*,*b*) = *w BIT c BIT b*

primrec
bin-succ-Pls: *bin-succ* (*Pls*) = *Pls BIT 1*
bin-succ-Min: *bin-succ* (*Min*) = *Pls*
bin-succ-BIT: *bin-succ* (*w BIT b*) = *cond*(*b*, *bin-succ*(*w*) *BIT 0*, *NCons*(*w*,*1*))

primrec
bin-pred-Pls: *bin-pred* (*Pls*) = *Min*
bin-pred-Min: *bin-pred* (*Min*) = *Min BIT 0*
bin-pred-BIT: *bin-pred* (*w BIT b*) = *cond*(*b*, *NCons*(*w*,*0*), *bin-pred*(*w*) *BIT 1*)

primrec
bin-minus-Pls:
bin-minus (*Pls*) = *Pls*
bin-minus-Min:
bin-minus (*Min*) = *Pls BIT 1*
bin-minus-BIT:
bin-minus (*w BIT b*) = *cond*(*b*, *bin-pred*(*NCons*(*bin-minus*(*w*),*0*)),
bin-minus(*w*) *BIT 0*)

primrec
bin-adder-Pls:
bin-adder (*Pls*) = (λ*w*∈*bin*. *w*)
bin-adder-Min:
bin-adder (*Min*) = (λ*w*∈*bin*. *bin-pred*(*w*))
bin-adder-BIT:
bin-adder (*v BIT x*) =
(λ*w*∈*bin*.
bin-case (*v BIT x*, *bin-pred*(*v BIT x*),
λ*w y*. *NCons*(*bin-adder* (*v*) ‘ *cond*(*x and y*, *bin-succ*(*w*), *w*),
x xor y),
w))

definition
bin-add :: [*i*,*i*]⇒*i* **where**
bin-add(*v*,*w*) ≡ *bin-adder*(*v*)‘*w*

primrec
bin-mult-Pls:
bin-mult (*Pls*,*w*) = *Pls*
bin-mult-Min:
bin-mult (*Min*,*w*) = *bin-minus*(*w*)
bin-mult-BIT:
bin-mult (*v BIT b*,*w*) = *cond*(*b*, *bin-add*(*NCons*(*bin-mult*(*v*,*w*),*0*),*w*),
NCons(*bin-mult*(*v*,*w*),*0*))

syntax
-Int0 :: *i* (‹#′ *0*›)
-Int1 :: *i* (‹#′ *1*›)
-Int2 :: *i* (‹#′ *2*›)
-Neg-Int1 :: *i* (‹#−′ *1*›)

-Neg-Int2 :: *i* (‹#−′ 2›)
translations
#0 ⇌ *CONST integ-of*(*CONST Pls*)
#1 ⇌ *CONST integ-of*(*CONST Pls BIT 1*)
#2 ⇌ *CONST integ-of*(*CONST Pls BIT 1 BIT 0*)
#−1 ⇌ *CONST integ-of*(*CONST Min*)
#−2 ⇌ *CONST integ-of*(*CONST Min BIT 0*)

syntax
-Int :: *num-token* ⇒ *i* (‹#-› *1000*)
-Neg-Int :: *num-token* ⇒ *i* (‹#−-› *1000*)

⟨*ML*⟩

declare *bin.intros* [*simp*,*TC*]

lemma *NCons-Pls-0*: *NCons*(*Pls*,*0*) = *Pls*
⟨*proof*⟩

lemma *NCons-Pls-1*: *NCons*(*Pls*,*1*) = *Pls BIT 1*
⟨*proof*⟩

lemma *NCons-Min-0*: *NCons*(*Min*,*0*) = *Min BIT 0*
⟨*proof*⟩

lemma *NCons-Min-1*: *NCons*(*Min*,*1*) = *Min*
⟨*proof*⟩

lemma *NCons-BIT*: *NCons*(*w BIT x*,*b*) = *w BIT x BIT b*
⟨*proof*⟩

lemmas *NCons-simps* [*simp*] =
NCons-Pls-0 NCons-Pls-1 NCons-Min-0 NCons-Min-1 NCons-BIT

lemma *integ-of-type* [*TC*]: *w* ∈ *bin* ⟹ *integ-of*(*w*) ∈ *int*
⟨*proof*⟩

lemma *NCons-type* [*TC*]: ⟦*w* ∈ *bin*; *b* ∈ *bool*⟧ ⟹ *NCons*(*w*,*b*) ∈ *bin*
⟨*proof*⟩

lemma *bin-succ-type* [*TC*]: *w* ∈ *bin* ⟹ *bin-succ*(*w*) ∈ *bin*
⟨*proof*⟩

lemma *bin-pred-type* [*TC*]: *w* ∈ *bin* ⟹ *bin-pred*(*w*) ∈ *bin*

⟨*proof*⟩

lemma *bin-minus-type* [*TC*]: $w \in bin \Longrightarrow bin\text{-}minus(w) \in bin$
⟨*proof*⟩

lemma *bin-add-type* [*rule-format*]:
$v \in bin \Longrightarrow \forall\, w{\in}bin.\ bin\text{-}add(v,w) \in bin$
⟨*proof*⟩

declare *bin-add-type* [*TC*]

lemma *bin-mult-type* [*TC*]: $[\![v \in bin;\ w \in bin]\!] \Longrightarrow bin\text{-}mult(v,w) \in bin$
⟨*proof*⟩

32.0.1 The Carry and Borrow Functions, *bin-succ* and *bin-pred*

lemma *integ-of-NCons* [*simp*]:
$[\![w \in bin;\ b \in bool]\!] \Longrightarrow integ\text{-}of(NCons(w,b)) = integ\text{-}of(w\ BIT\ b)$
⟨*proof*⟩

lemma *integ-of-succ* [*simp*]:
$w \in bin \Longrightarrow integ\text{-}of(bin\text{-}succ(w)) = \$\#1\ \$+\ integ\text{-}of(w)$
⟨*proof*⟩

lemma *integ-of-pred* [*simp*]:
$w \in bin \Longrightarrow integ\text{-}of(bin\text{-}pred(w)) = \$-\ (\$\#1)\ \$+\ integ\text{-}of(w)$
⟨*proof*⟩

32.0.2 *bin-minus*: Unary Negation of Binary Integers

lemma *integ-of-minus*: $w \in bin \Longrightarrow integ\text{-}of(bin\text{-}minus(w)) = \$-\ integ\text{-}of(w)$
⟨*proof*⟩

32.0.3 *bin-add*: Binary Addition

lemma *bin-add-Pls* [*simp*]: $w \in bin \Longrightarrow bin\text{-}add(Pls,w) = w$
⟨*proof*⟩

lemma *bin-add-Pls-right*: $w \in bin \Longrightarrow bin\text{-}add(w,Pls) = w$
⟨*proof*⟩

lemma *bin-add-Min* [*simp*]: $w \in bin \Longrightarrow bin\text{-}add(Min,w) = bin\text{-}pred(w)$
⟨*proof*⟩

lemma *bin-add-Min-right*: $w \in bin \Longrightarrow bin\text{-}add(w,Min) = bin\text{-}pred(w)$
⟨*proof*⟩

lemma *bin-add-BIT-Pls* [*simp*]: $bin\text{-}add(v\ BIT\ x,Pls) = v\ BIT\ x$
⟨*proof*⟩

lemma *bin-add-BIT-Min* [*simp*]: *bin-add*(*v BIT x*,*Min*) = *bin-pred*(*v BIT x*)
⟨*proof*⟩

lemma *bin-add-BIT-BIT* [*simp*]:
⟦*w* ∈ *bin*; *y* ∈ *bool*⟧
⟹ *bin-add*(*v BIT x*, *w BIT y*) =
NCons(*bin-add*(*v*, *cond*(*x and y*, *bin-succ*(*w*), *w*)), *x xor y*)
⟨*proof*⟩

lemma *integ-of-add* [*rule-format*]:
v ∈ *bin* ⟹
∀ *w*∈*bin*. *integ-of*(*bin-add*(*v*,*w*)) = *integ-of*(*v*) $+ *integ-of*(*w*)
⟨*proof*⟩

lemma *diff-integ-of-eq*:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*v*) $− *integ-of*(*w*) = *integ-of*(*bin-add* (*v*, *bin-minus*(*w*)))
⟨*proof*⟩

32.0.4 *bin-mult*: Binary Multiplication

lemma *integ-of-mult*:
⟦*v* ∈ *bin*; *w* ∈ *bin*⟧
⟹ *integ-of*(*bin-mult*(*v*,*w*)) = *integ-of*(*v*) $∗ *integ-of*(*w*)
⟨*proof*⟩

32.1 Computations

lemma *bin-succ-1*: *bin-succ*(*w BIT 1*) = *bin-succ*(*w*) *BIT 0*
⟨*proof*⟩

lemma *bin-succ-0*: *bin-succ*(*w BIT 0*) = *NCons*(*w*,*1*)
⟨*proof*⟩

lemma *bin-pred-1*: *bin-pred*(*w BIT 1*) = *NCons*(*w*,*0*)
⟨*proof*⟩

lemma *bin-pred-0*: *bin-pred*(*w BIT 0*) = *bin-pred*(*w*) *BIT 1*
⟨*proof*⟩

lemma *bin-minus-1*: *bin-minus*(*w BIT 1*) = *bin-pred*(*NCons*(*bin-minus*(*w*), *0*))
⟨*proof*⟩

lemma *bin-minus-0*: *bin-minus*(*w BIT 0*) = *bin-minus*(*w*) *BIT 0*
⟨*proof*⟩

lemma *bin-add-BIT-11*: $w \in bin \Longrightarrow bin\text{-}add(v\ BIT\ 1,\ w\ BIT\ 1) =$
$NCons(bin\text{-}add(v,\ bin\text{-}succ(w)),\ 0)$
⟨*proof*⟩

lemma *bin-add-BIT-10*: $w \in bin \Longrightarrow bin\text{-}add(v\ BIT\ 1,\ w\ BIT\ 0) =$
$NCons(bin\text{-}add(v,w),\ 1)$
⟨*proof*⟩

lemma *bin-add-BIT-0*: $[\![w \in bin;\ \ y \in bool]\!]$
$\Longrightarrow bin\text{-}add(v\ BIT\ 0,\ w\ BIT\ y) = NCons(bin\text{-}add(v,w),\ y)$
⟨*proof*⟩

lemma *bin-mult-1*: $bin\text{-}mult(v\ BIT\ 1,\ w) = bin\text{-}add(NCons(bin\text{-}mult(v,w),0),\ w)$
⟨*proof*⟩

lemma *bin-mult-0*: $bin\text{-}mult(v\ BIT\ 0,\ w) = NCons(bin\text{-}mult(v,w),0)$
⟨*proof*⟩

lemma *int-of-0*: $\$\#0 = \#0$
⟨*proof*⟩

lemma *int-of-succ*: $\$\#\ succ(n) = \#1\ \$+\ \$\#n$
⟨*proof*⟩

lemma *zminus-0* [*simp*]: $\$-\ \#0 = \#0$
⟨*proof*⟩

lemma *zadd-0-intify* [*simp*]: $\#0\ \$+\ z = intify(z)$
⟨*proof*⟩

lemma *zadd-0-right-intify* [*simp*]: $z\ \$+\ \#0 = intify(z)$
⟨*proof*⟩

lemma *zmult-1-intify* [*simp*]: $\#1\ \$*\ z = intify(z)$
⟨*proof*⟩

lemma *zmult-1-right-intify* [*simp*]: $z\ \$*\ \#1 = intify(z)$
⟨*proof*⟩

lemma *zmult-0* [*simp*]: $\#0\ \$*\ z = \#0$
⟨*proof*⟩

lemma *zmult-0-right* [*simp*]: $z \$* \#0 = \#0$
⟨*proof*⟩

lemma *zmult-minus1* [*simp*]: $\#-1 \$* z = \$-z$
⟨*proof*⟩

lemma *zmult-minus1-right* [*simp*]: $z \$* \#-1 = \$-z$
⟨*proof*⟩

32.2 Simplification Rules for Comparison of Binary Numbers

Thanks to Norbert Voelker

lemma *eq-integ-of-eq*:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow ((integ\text{-}of(v)) = integ\text{-}of(w)) \longleftrightarrow$
$iszero\ (integ\text{-}of\ (bin\text{-}add\ (v,\ bin\text{-}minus(w))))$
⟨*proof*⟩

lemma *iszero-integ-of-Pls*: $iszero\ (integ\text{-}of(Pls))$
⟨*proof*⟩

lemma *nonzero-integ-of-Min*: $\neg\ iszero\ (integ\text{-}of(Min))$
⟨*proof*⟩

lemma *iszero-integ-of-BIT*:
$[\![w \in bin;\ x \in bool]\!]$
$\Longrightarrow iszero\ (integ\text{-}of\ (w\ BIT\ x)) \longleftrightarrow (x=0 \wedge iszero\ (integ\text{-}of(w)))$
⟨*proof*⟩

lemma *iszero-integ-of-0*:
$w \in bin \Longrightarrow iszero\ (integ\text{-}of\ (w\ BIT\ 0)) \longleftrightarrow iszero\ (integ\text{-}of(w))$
⟨*proof*⟩

lemma *iszero-integ-of-1*: $w \in bin \Longrightarrow \neg\ iszero\ (integ\text{-}of\ (w\ BIT\ 1))$
⟨*proof*⟩

lemma *less-integ-of-eq-neg*:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow integ\text{-}of(v)\ \$< integ\text{-}of(w)$
$\longleftrightarrow znegative\ (integ\text{-}of\ (bin\text{-}add\ (v,\ bin\text{-}minus(w))))$
⟨*proof*⟩

lemma *not-neg-integ-of-Pls*: $\neg\ znegative\ (integ\text{-}of(Pls))$

⟨*proof*⟩

lemma *neg-integ-of-Min*: *znegative* (*integ-of*(*Min*))
⟨*proof*⟩

lemma *neg-integ-of-BIT*:
⟦*w* ∈ *bin*; *x* ∈ *bool*⟧
⟹ *znegative* (*integ-of* (*w BIT x*)) ⟷ *znegative* (*integ-of*(*w*))
⟨*proof*⟩

lemma *le-integ-of-eq-not-less*:
(*integ-of*(*x*) $≤ (*integ-of*(*w*))) ⟷ ¬ (*integ-of*(*w*) $< (*integ-of*(*x*)))
⟨*proof*⟩

declare *bin-succ-BIT* [*simp del*]
bin-pred-BIT [*simp del*]
bin-minus-BIT [*simp del*]
NCons-Pls [*simp del*]
NCons-Min [*simp del*]
bin-adder-BIT [*simp del*]
bin-mult-BIT [*simp del*]

declare *integ-of-Pls* [*simp del*] *integ-of-Min* [*simp del*] *integ-of-BIT* [*simp del*]

lemmas *bin-arith-extra-simps* =
integ-of-add [*symmetric*]
integ-of-minus [*symmetric*]
integ-of-mult [*symmetric*]
bin-succ-1 bin-succ-0
bin-pred-1 bin-pred-0
bin-minus-1 bin-minus-0
bin-add-Pls-right bin-add-Min-right
bin-add-BIT-0 bin-add-BIT-10 bin-add-BIT-11
diff-integ-of-eq
bin-mult-1 bin-mult-0 NCons-simps

lemmas *bin-arith-simps* =
bin-pred-Pls bin-pred-Min
bin-succ-Pls bin-succ-Min
bin-add-Pls bin-add-Min
bin-minus-Pls bin-minus-Min

bin-mult-Pls bin-mult-Min
bin-arith-extra-simps

lemmas *bin-rel-simps* =
eq-integ-of-eq iszero-integ-of-Pls nonzero-integ-of-Min
iszero-integ-of-0 iszero-integ-of-1
less-integ-of-eq-neg
not-neg-integ-of-Pls neg-integ-of-Min neg-integ-of-BIT
le-integ-of-eq-not-less

declare *bin-arith-simps* [*simp*]
declare *bin-rel-simps* [*simp*]

lemma *add-integ-of-left* [*simp*]:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow$ *integ-of*(v) \$+ (*integ-of*(w) \$+ z) = (*integ-of*(*bin-add*(v,w)) \$+ z)
⟨*proof*⟩

lemma *mult-integ-of-left* [*simp*]:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow$ *integ-of*(v) \$∗ (*integ-of*(w) \$∗ z) = (*integ-of*(*bin-mult*(v,w)) \$∗ z)
⟨*proof*⟩

lemma *add-integ-of-diff1* [*simp*]:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow$ *integ-of*(v) \$+ (*integ-of*(w) \$− c) = *integ-of*(*bin-add*(v,w)) \$− (c)
⟨*proof*⟩

lemma *add-integ-of-diff2* [*simp*]:
$[\![v \in bin;\ w \in bin]\!]$
$\Longrightarrow$ *integ-of*(v) \$+ (c \$− *integ-of*(w)) =
integ-of (*bin-add* (v, *bin-minus*(w))) \$+ (c)
⟨*proof*⟩

declare *int-of-0* [*simp*] *int-of-succ* [*simp*]

lemma *zdiff0* [*simp*]: #0 \$− x = \$−x
⟨*proof*⟩

lemma *zdiff0-right* [*simp*]: x \$− #0 = *intify*(x)
⟨*proof*⟩

lemma *zdiff-self* [*simp*]: $x \ \$- \ x = \#0$
⟨*proof*⟩

lemma *znegative-iff-zless-0*: $k \in int \Longrightarrow znegative(k) \longleftrightarrow k \ \$< \ \#0$
⟨*proof*⟩

lemma *zero-zless-imp-znegative-zminus*: $[\![\#0 \ \$< \ k; \ k \in int]\!] \Longrightarrow znegative(\$-k)$
⟨*proof*⟩

lemma *zero-zle-int-of* [*simp*]: $\#0 \ \$\leq \ \$\# \ n$
⟨*proof*⟩

lemma *nat-of-0* [*simp*]: $nat\text{-}of(\#0) = 0$
⟨*proof*⟩

lemma *nat-le-int0-lemma*: $[\![z \ \$\leq \ \$\#0; \ z \in int]\!] \Longrightarrow nat\text{-}of(z) = 0$
⟨*proof*⟩

lemma *nat-le-int0*: $z \ \$\leq \ \$\#0 \Longrightarrow nat\text{-}of(z) = 0$
⟨*proof*⟩

lemma *int-of-eq-0-imp-natify-eq-0*: $\$\# \ n = \#0 \Longrightarrow natify(n) = 0$
⟨*proof*⟩

lemma *nat-of-zminus-int-of*: $nat\text{-}of(\$- \ \$\# \ n) = 0$
⟨*proof*⟩

lemma *int-of-nat-of*: $\#0 \ \$\leq \ z \Longrightarrow \$\# \ nat\text{-}of(z) = intify(z)$
⟨*proof*⟩

declare *int-of-nat-of* [*simp*] *nat-of-zminus-int-of* [*simp*]

lemma *int-of-nat-of-if*: $\$\# \ nat\text{-}of(z) = (if \ \#0 \ \$\leq \ z \ then \ intify(z) \ else \ \#0)$
⟨*proof*⟩

lemma *zless-nat-iff-int-zless*: $[\![m \in nat; \ z \in int]\!] \Longrightarrow (m < nat\text{-}of(z)) \longleftrightarrow (\$\#m \ \$< \ z)$
⟨*proof*⟩

lemma *zless-nat-conj-lemma*: $\$\#0 \ \$< \ z \Longrightarrow (nat\text{-}of(w) < nat\text{-}of(z)) \longleftrightarrow (w \ \$< \ z)$
⟨*proof*⟩

lemma *zless-nat-conj*: $(nat\text{-}of(w) < nat\text{-}of(z)) \longleftrightarrow (\$\#0 \ \$< \ z \wedge w \ \$< \ z)$
⟨*proof*⟩

lemma *integ-of-minus-reorient* [*simp*]:
(*integ-of*(*w*) = $− *x*) ⟷ ($− *x* = *integ-of*(*w*))
⟨*proof*⟩

lemma *integ-of-add-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $+ *y*) ⟷ (*x* $+ *y* = *integ-of*(*w*))
⟨*proof*⟩

lemma *integ-of-diff-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $− *y*) ⟷ (*x* $− *y* = *integ-of*(*w*))
⟨*proof*⟩

lemma *integ-of-mult-reorient* [*simp*]:
(*integ-of*(*w*) = *x* $∗ *y*) ⟷ (*x* $∗ *y* = *integ-of*(*w*))
⟨*proof*⟩

lemmas [*simp*] =
zminus-equation [**where** *y* = *integ-of*(*w*)]
equation-zminus [**where** *x* = *integ-of*(*w*)]
for *w*

lemmas [*iff*] =
zminus-zless [**where** *y* = *integ-of*(*w*)]
zless-zminus [**where** *x* = *integ-of*(*w*)]
for *w*

lemmas [*iff*] =
zminus-zle [**where** *y* = *integ-of*(*w*)]
zle-zminus [**where** *x* = *integ-of*(*w*)]
for *w*

lemmas [*simp*] =
Let-def [**where** *s* = *integ-of*(*w*)] **for** *w*

lemma *zless-iff-zdiff-zless-0*: (*x* $< *y*) ⟷ (*x*$−*y* $< #0)
⟨*proof*⟩

lemma *eq-iff-zdiff-eq-0*: ⟦*x* ∈ *int*; *y* ∈ *int*⟧ ⟹ (*x* = *y*) ⟷ (*x*$−*y* = #0)
⟨*proof*⟩

lemma *zle-iff-zdiff-zle-0*: $(x\ \$\leq\ y) \longleftrightarrow (x\$-y\ \$\leq\ \#0)$
⟨*proof*⟩

lemma *left-zadd-zmult-distrib*: $i\$*u\ \$+\ (j\$*u\ \$+\ k) = (i\$+j)\$*u\ \$+\ k$
⟨*proof*⟩

lemma *eq-add-iff1*: $(i\$*u\ \$+\ m = j\$*u\ \$+\ n) \longleftrightarrow ((i\$-j)\$*u\ \$+\ m = intify(n))$
⟨*proof*⟩

lemma *eq-add-iff2*: $(i\$*u\ \$+\ m = j\$*u\ \$+\ n) \longleftrightarrow (intify(m) = (j\$-i)\$*u\ \$+\ n)$
⟨*proof*⟩

context fixes $n :: i$
begin

lemmas *rel-iff-rel-0-rls* =
zless-iff-zdiff-zless-0 [**where** $y = u\ \$+\ v$]
eq-iff-zdiff-eq-0 [**where** $y = u\ \$+\ v$]
zle-iff-zdiff-zle-0 [**where** $y = u\ \$+\ v$]
zless-iff-zdiff-zless-0 [**where** $y = n$]
eq-iff-zdiff-eq-0 [**where** $y = n$]
zle-iff-zdiff-zle-0 [**where** $y = n$]
for $u\ v$

lemma *less-add-iff1*: $(i\$*u\ \$+\ m\ \$<\ j\$*u\ \$+\ n) \longleftrightarrow ((i\$-j)\$*u\ \$+\ m\ \$<\ n)$
⟨*proof*⟩

lemma *less-add-iff2*: $(i\$*u\ \$+\ m\ \$<\ j\$*u\ \$+\ n) \longleftrightarrow (m\ \$<\ (j\$-i)\$*u\ \$+\ n)$
⟨*proof*⟩

end

lemma *le-add-iff1*: $(i\$*u\ \$+\ m\ \$\leq\ j\$*u\ \$+\ n) \longleftrightarrow ((i\$-j)\$*u\ \$+\ m\ \$\leq\ n)$
⟨*proof*⟩

lemma *le-add-iff2*: $(i\$*u\ \$+\ m\ \$\leq\ j\$*u\ \$+\ n) \longleftrightarrow (m\ \$\leq\ (j\$-i)\$*u\ \$+\ n)$
⟨*proof*⟩

⟨*ML*⟩

32.3 examples:

combine-numerals-prod (products of separate literals)

lemma *#5 $* x $* #3 = y ⟨proof⟩*

schematic-goal *y2 $+ ?x42 = y $+ y2 ⟨proof⟩*

lemma *oo : int ⟹ l $+ (l $+ #2) $+ oo = oo ⟨proof⟩*

lemma *#9$*x $+ y = x$*#23 $+ z ⟨proof⟩*
lemma *y $+ x = x $+ z ⟨proof⟩*

lemma *x : int ⟹ x $+ y $+ z = x $+ z ⟨proof⟩*
lemma *x : int ⟹ y $+ (z $+ x) = z $+ x ⟨proof⟩*
lemma *z : int ⟹ x $+ y $+ z = (z $+ y) $+ (x $+ w) ⟨proof⟩*
lemma *z : int ⟹ x$*y $+ z = (z $+ y) $+ (y$*x $+ w) ⟨proof⟩*

lemma *#−3 $* x $+ y $≤ x $* #2 $+ z ⟨proof⟩*
lemma *y $+ x $≤ x $+ z ⟨proof⟩*
lemma *x $+ y $+ z $≤ x $+ z ⟨proof⟩*

lemma *y $+ (z $+ x) $< z $+ x ⟨proof⟩*
lemma *x $+ y $+ z $< (z $+ y) $+ (x $+ w) ⟨proof⟩*
lemma *x$*y $+ z $< (z $+ y) $+ (y$*x $+ w) ⟨proof⟩*

lemma *l $+ #2 $+ #2 $+ #2 $+ (l $+ #2) $+ (oo $+ #2) = uu ⟨proof⟩*
lemma *u : int ⟹ #2 $* u = u ⟨proof⟩*
lemma *(i $+ j $+ #12 $+ k) $− #15 = y ⟨proof⟩*
lemma *(i $+ j $+ #12 $+ k) $− #5 = y ⟨proof⟩*

lemma *y $− b $< b ⟨proof⟩*
lemma *y $− (#3 $* b $+ c) $< b $− #2 $* c ⟨proof⟩*

lemma *(#2 $* x $− (u $* v) $+ y) $− v $* #3 $* u = w ⟨proof⟩*
lemma *(#2 $* x $* u $* v $+ (u $* v) $* #4 $+ y) $− v $* u $* #4 = w ⟨proof⟩*
lemma *(#2 $* x $* u $* v $+ (u $* v) $* #4 $+ y) $− v $* u = w ⟨proof⟩*
lemma *u $* v $− (x $* u $* v $+ (u $* v) $* #4 $+ y) = w ⟨proof⟩*

lemma *(i $+ j $+ #12 $+ k) = u $+ #15 $+ y ⟨proof⟩*
lemma *(i $+ j $* #2 $+ #12 $+ k) = j $+ #5 $+ y ⟨proof⟩*

lemma *#2 $* y $+ #3 $* z $+ #6 $* w $+ #2 $* y $+ #3 $* z $+ #2 $* u = #2 $* y′ $+ #3 $* z′ $+ #6 $* w′ $+ #2 $* y′ $+ #3 $* z′ $+ u $+ vv ⟨proof⟩*

lemma *a $+ $−(b$+c) $+ b = d ⟨proof⟩*
lemma *a $+ $−(b$+c) $− b = d ⟨proof⟩*

negative numerals

lemma (*i* $+ *j* $+ #−*2* $+ *k*) $− (*u* $+ #*5* $+ *y*) = *zz* ⟨*proof*⟩
lemma (*i* $+ *j* $+ #−*3* $+ *k*) $< *u* $+ #*5* $+ *y* ⟨*proof*⟩
lemma (*i* $+ *j* $+ #*3* $+ *k*) $< *u* $+ #−*6* $+ *y* ⟨*proof*⟩
lemma (*i* $+ *j* $+ #−*12* $+ *k*) $− #*15* = *y* ⟨*proof*⟩
lemma (*i* $+ *j* $+ #*12* $+ *k*) $− #−*15* = *y* ⟨*proof*⟩
lemma (*i* $+ *j* $+ #−*12* $+ *k*) $− #−*15* = *y* ⟨*proof*⟩

Multiplying separated numerals

lemma #*6* $∗ ($# *x* $∗ #*2*) = *uu* ⟨*proof*⟩
lemma #*4* $∗ ($# *x* $∗ $# *x*) $∗ (#*2* $∗ $# *x*) = *uu* ⟨*proof*⟩

end

33 The Division Operators Div and Mod

theory *IntDiv*
imports *Bin OrderArith*
begin

definition
quorem :: [*i*,*i*] ⇒ *o* **where**
quorem ≡ λ⟨*a*,*b*⟩ ⟨*q*,*r*⟩.
a = *b*$∗*q* $+ *r* ∧
(#*0*$<*b* ∧ #*0*$≤*r* ∧ *r*$<*b* | ¬(#*0*$<*b*) ∧ *b*$<*r* ∧ *r* $≤ #*0*)

definition
adjust :: [*i*,*i*] ⇒ *i* **where**
adjust(*b*) ≡ λ⟨*q*,*r*⟩. *if* #*0* $≤ *r*$−*b* *then* <#*2*$∗*q* $+ #*1*,*r*$−*b*>
else <#*2*$∗*q*,*r*>

definition
posDivAlg :: *i* ⇒ *i* **where**

posDivAlg(*ab*) ≡
wfrec(*measure*(*int*∗*int*, λ⟨*a*,*b*⟩. *nat-of* (*a* $− *b* $+ #*1*)),
ab,
λ⟨*a*,*b*⟩ *f*. *if* (*a*$<*b* | *b*$≤#*0*) *then* <#*0*,*a*>
else *adjust*(*b*, *f* ‘ <*a*,#*2*$∗*b*>))

definition
negDivAlg :: *i* ⇒ *i* **where**

```
  negDivAlg(ab) ≡
    wfrec(measure(int*int, λ⟨a,b⟩. nat-of ($− a $− b)),
          ab,
          λ⟨a,b⟩ f. if (#0 $≤ a$+b | b$≤#0) then <#−1,a$+b>
                    else adjust(b, f ‘ <a,#2$*b>))
```

definition

```
  negateSnd :: i ⇒ i  where
    negateSnd ≡ λ⟨q,r⟩. <q, $−r>
```

definition

```
  divAlg :: i ⇒ i  where
    divAlg ≡
      λ⟨a,b⟩. if #0 $≤ a then
                if #0 $≤ b then posDivAlg (⟨a,b⟩)
                else if a=#0 then <#0,#0>
                     else negateSnd (negDivAlg (<$−a,$−b>))
              else
                if #0$<b then negDivAlg (⟨a,b⟩)
                else          negateSnd (posDivAlg (<$−a,$−b>))
```

definition

```
  zdiv  :: [i,i]⇒i                    (infixl ‹zdiv› 70)  where
    a zdiv b ≡ fst (divAlg (<intify(a), intify(b)>))
```

definition

```
  zmod  :: [i,i]⇒i                    (infixl ‹zmod› 70)  where
    a zmod b ≡ snd (divAlg (<intify(a), intify(b)>))
```

lemma *zspos-add-zspos-imp-zspos*: $[\![\#0\ \$< x;\ \#0\ \$< y]\!] \Longrightarrow \#0\ \$< x\ \$+\ y$
⟨*proof*⟩

lemma *zpos-add-zpos-imp-zpos*: $[\![\#0\ \$\leq x;\ \#0\ \$\leq y]\!] \Longrightarrow \#0\ \$\leq x\ \$+\ y$
⟨*proof*⟩

lemma *zneg-add-zneg-imp-zneg*: $[\![x\ \$< \#0;\ y\ \$< \#0]\!] \Longrightarrow x\ \$+\ y\ \$< \#0$
⟨*proof*⟩

lemma *zneg-or-0-add-zneg-or-0-imp-zneg-or-0*:
$[\![x\ \$\leq \#0;\ y\ \$\leq \#0]\!] \Longrightarrow x\ \$+\ y\ \$\leq \#0$
⟨*proof*⟩

lemma *zero-lt-zmagnitude*: $[\![\#0 \ \$< k;\ k \in int]\!] \Longrightarrow 0 < zmagnitude(k)$
⟨*proof*⟩

lemma *zless-add-succ-iff*:
$(w \ \$< z \ \$+ \ \$\# \ succ(m)) \longleftrightarrow (w \ \$< z \ \$+ \ \$\#m \mid intify(w) = z \ \$+ \ \$\#m)$
⟨*proof*⟩

lemma *zadd-succ-lemma*:
$z \in int \Longrightarrow (w \ \$+ \ \$\# \ succ(m) \ \$\leq z) \longleftrightarrow (w \ \$+ \ \$\#m \ \$< z)$
⟨*proof*⟩

lemma *zadd-succ-zle-iff*: $(w \ \$+ \ \$\# \ succ(m) \ \$\leq z) \longleftrightarrow (w \ \$+ \ \$\#m \ \$< z)$
⟨*proof*⟩

lemma *zless-add1-iff-zle*: $(w \ \$< z \ \$+ \ \#1) \longleftrightarrow (w \$\leq z)$
⟨*proof*⟩

lemma *add1-zle-iff*: $(w \ \$+ \ \#1 \ \$\leq z) \longleftrightarrow (w \ \$< z)$
⟨*proof*⟩

lemma *add1-left-zle-iff*: $(\#1 \ \$+ \ w \ \$\leq z) \longleftrightarrow (w \ \$< z)$
⟨*proof*⟩

lemma *zmult-mono-lemma*: $k \in nat \Longrightarrow i \ \$\leq j \Longrightarrow i \ \$* \ \$\#k \ \$\leq j \ \$* \ \$\#k$
⟨*proof*⟩

lemma *zmult-zle-mono1*: $[\![i \ \$\leq j;\ \ \#0 \ \$\leq k]\!] \Longrightarrow i\$*k \ \$\leq j\$*k$
⟨*proof*⟩

lemma *zmult-zle-mono1-neg*: $[\![i \ \$\leq j;\ \ k \ \$\leq \#0]\!] \Longrightarrow j\$*k \ \$\leq i\$*k$
⟨*proof*⟩

lemma *zmult-zle-mono2*: $[\![i \ \$\leq j;\ \ \#0 \ \$\leq k]\!] \Longrightarrow k\$*i \ \$\leq k\$*j$
⟨*proof*⟩

lemma *zmult-zle-mono2-neg*: $[\![i \ \$\leq j;\ \ k \ \$\leq \#0]\!] \Longrightarrow k\$*j \ \$\leq k\$*i$
⟨*proof*⟩

lemma *zmult-zle-mono*:

$[\![i \$\leq j;\ k \$\leq l;\ \#0 \$\leq j;\ \#0 \$\leq k]\!] \Longrightarrow i\$*k \$\leq j\$*l$
⟨*proof*⟩

lemma *zmult-zless-mono2-lemma* [*rule-format*]:
$[\![i\$<j;\ k \in nat]\!] \Longrightarrow 0<k \longrightarrow \$\#k \$* i \$< \$\#k \$* j$
⟨*proof*⟩

lemma *zmult-zless-mono2*: $[\![i\$<j;\ \#0 \$< k]\!] \Longrightarrow k\$*i \$< k\$*j$
⟨*proof*⟩

lemma *zmult-zless-mono1*: $[\![i\$<j;\ \#0 \$< k]\!] \Longrightarrow i\$*k \$< j\$*k$
⟨*proof*⟩

lemma *zmult-zless-mono*:
$[\![i \$< j;\ k \$< l;\ \#0 \$< j;\ \#0 \$< k]\!] \Longrightarrow i\$*k \$< j\$*l$
⟨*proof*⟩

lemma *zmult-zless-mono1-neg*: $[\![i \$< j;\ k \$< \#0]\!] \Longrightarrow j\$*k \$< i\$*k$
⟨*proof*⟩

lemma *zmult-zless-mono2-neg*: $[\![i \$< j;\ k \$< \#0]\!] \Longrightarrow k\$*j \$< k\$*i$
⟨*proof*⟩

lemma *zmult-eq-lemma*:
$[\![m \in int;\ n \in int]\!] \Longrightarrow (m = \#0 \mid n = \#0) \longleftrightarrow (m\$*n = \#0)$
⟨*proof*⟩

lemma *zmult-eq-0-iff* [*iff*]: $(m\$*n = \#0) \longleftrightarrow (intify(m) = \#0 \mid intify(n) = \#0)$
⟨*proof*⟩

lemma *zmult-zless-lemma*:
$[\![k \in int;\ m \in int;\ n \in int]\!]$
$\Longrightarrow (m\$*k \$< n\$*k) \longleftrightarrow ((\#0 \$< k \wedge m\$<n) \mid (k \$< \#0 \wedge n\$<m))$
⟨*proof*⟩

lemma *zmult-zless-cancel2*:
$(m\$*k \$< n\$*k) \longleftrightarrow ((\#0 \$< k \wedge m\$<n) \mid (k \$< \#0 \wedge n\$<m))$
⟨*proof*⟩

lemma *zmult-zless-cancel1*:
$(k\$*m\ \$< k\$*n) \longleftrightarrow ((\#0\ \$< k \wedge m\$<n) \mid (k\ \$< \#0 \wedge n\$<m))$
⟨*proof*⟩

lemma *zmult-zle-cancel2*:
$(m\$*k\ \$\leq n\$*k) \longleftrightarrow ((\#0\ \$< k \longrightarrow m\$\leq n) \wedge (k\ \$< \#0 \longrightarrow n\$\leq m))$
⟨*proof*⟩

lemma *zmult-zle-cancel1*:
$(k\$*m\ \$\leq k\$*n) \longleftrightarrow ((\#0\ \$< k \longrightarrow m\$\leq n) \wedge (k\ \$< \#0 \longrightarrow n\$\leq m))$
⟨*proof*⟩

lemma *int-eq-iff-zle*: $[\![m \in int;\ n \in int]\!] \Longrightarrow m{=}n \longleftrightarrow (m\ \$\leq n \wedge n\ \$\leq m)$
⟨*proof*⟩

lemma *zmult-cancel2-lemma*:
$[\![k \in int;\ m \in int;\ n \in int]\!] \Longrightarrow (m\$*k = n\$*k) \longleftrightarrow (k{=}\#0 \mid m{=}n)$
⟨*proof*⟩

lemma *zmult-cancel2* [*simp*]:
$(m\$*k = n\$*k) \longleftrightarrow (intify(k) = \#0 \mid intify(m) = intify(n))$
⟨*proof*⟩

lemma *zmult-cancel1* [*simp*]:
$(k\$*m = k\$*n) \longleftrightarrow (intify(k) = \#0 \mid intify(m) = intify(n))$
⟨*proof*⟩

33.1 Uniqueness and monotonicity of quotients and remainders

lemma *unique-quotient-lemma*:
$[\![b\$*q'\ \$+ r'\ \$\leq b\$*q\ \$+ r;\ \ \#0\ \$\leq r';\ \ \#0\ \$< b;\ \ r\ \$< b]\!]$
$\Longrightarrow q'\ \$\leq q$
⟨*proof*⟩

lemma *unique-quotient-lemma-neg*:
$[\![b\$*q'\ \$+ r'\ \$\leq b\$*q\ \$+ r;\ \ r\ \$\leq \#0;\ \ b\ \$< \#0;\ \ b\ \$< r']\!]$
$\Longrightarrow q\ \$\leq q'$
⟨*proof*⟩

lemma *unique-quotient*:
$[\![quorem\ (\langle a,b\rangle, \langle q,r\rangle);\ \ quorem\ (\langle a,b\rangle, <q',r'>);\ \ b \in int;\ b \neq \#0;$
$q \in int;\ q' \in int]\!] \Longrightarrow q = q'$
⟨*proof*⟩

lemma *unique-remainder*:
$[\![quorem\ (\langle a,b\rangle, \langle q,r\rangle);\ \ quorem\ (\langle a,b\rangle, <q',r'>);\ \ b \in int;\ b \neq \#0;$
$q \in int;\ q' \in int;$

$r \in int;\ r' \in int\rrbracket \Longrightarrow r = r'$
⟨*proof*⟩

33.2 Correctness of posDivAlg, the Division Algorithm for $a \geq 0$ and $b > 0$

lemma *adjust-eq* [*simp*]:
$adjust(b, \langle q,r\rangle) = (let\ diff = r\$-b\ in$
$\quad if\ \#0\ \$\leq\ diff\ then\ <\#2\$*q\ \$+\ \#1,diff>$
$\quad\quad else\ <\#2\$*q,r>)$
⟨*proof*⟩

lemma *posDivAlg-termination*:
$\llbracket \#0\ \$<\ b;\ \neg\ a\ \$<\ b\rrbracket$
$\Longrightarrow nat\text{-}of(a\ \$-\ \#2\ \$*\ b\ \$+\ \#1) < nat\text{-}of(a\ \$-\ b\ \$+\ \#1)$
⟨*proof*⟩

lemmas *posDivAlg-unfold* = *def-wfrec* [*OF posDivAlg-def wf-measure*]

lemma *posDivAlg-eqn*:
$\llbracket \#0\ \$<\ b;\ a \in int;\ b \in int\rrbracket \Longrightarrow$
$posDivAlg(\langle a,b\rangle) =$
$(if\ a\$<b\ then\ <\#0,a>\ else\ adjust(b,\ posDivAlg\ (<a,\ \#2\$*b>)))$
⟨*proof*⟩

lemma *posDivAlg-induct-lemma* [*rule-format*]:
assumes *prem*:
$\bigwedge a\ b.\ \llbracket a \in int;\ b \in int;$
$\neg\ (a\ \$<\ b \mid b\ \$\leq\ \#0) \longrightarrow P(<a,\ \#2\ \$*\ b>)\rrbracket \Longrightarrow P(\langle a,b\rangle)$
shows $\langle u,v\rangle \in int*int \Longrightarrow P(\langle u,v\rangle)$
⟨*proof*⟩

lemma *posDivAlg-induct* [*consumes 2*]:
assumes *u-int*: $u \in int$
and *v-int*: $v \in int$
and *ih*: $\bigwedge a\ b.\ \llbracket a \in int;\ b \in int;$
$\neg\ (a\ \$<\ b \mid b\ \$\leq\ \#0) \longrightarrow P(a,\ \#2\ \$*\ b)\rrbracket \Longrightarrow P(a,b)$
shows $P(u,v)$
⟨*proof*⟩

lemma *intify-eq-0-iff-zle*: $intify(m) = \#0 \longleftrightarrow (m\ \$\leq\ \#0 \wedge \#0\ \$\leq\ m)$
⟨*proof*⟩

33.3 Some convenient biconditionals for products of signs

lemma *zmult-pos*: $\llbracket \#0\ \$<\ i;\ \#0\ \$<\ j\rrbracket \Longrightarrow \#0\ \$<\ i\ \$*\ j$

⟨*proof*⟩

lemma *zmult-neg*: ⟦*i* $< #0; *j* $< #0⟧ ⟹ #0 $< *i* $∗ *j*
⟨*proof*⟩

lemma *zmult-pos-neg*: ⟦#0 $< *i*; *j* $< #0⟧ ⟹ *i* $∗ *j* $< #0
⟨*proof*⟩

lemma *int-0-less-lemma*:
⟦*x* ∈ *int*; *y* ∈ *int*⟧
⟹ (#0 $< *x* $∗ *y*) ⟷ (#0 $< *x* ∧ #0 $< *y* | *x* $< #0 ∧ *y* $< #0)
⟨*proof*⟩

lemma *int-0-less-mult-iff*:
(#0 $< *x* $∗ *y*) ⟷ (#0 $< *x* ∧ #0 $< *y* | *x* $< #0 ∧ *y* $< #0)
⟨*proof*⟩

lemma *int-0-le-lemma*:
⟦*x* ∈ *int*; *y* ∈ *int*⟧
⟹ (#0 $≤ *x* $∗ *y*) ⟷ (#0 $≤ *x* ∧ #0 $≤ *y* | *x* $≤ #0 ∧ *y* $≤ #0)
⟨*proof*⟩

lemma *int-0-le-mult-iff*:
(#0 $≤ *x* $∗ *y*) ⟷ ((#0 $≤ *x* ∧ #0 $≤ *y*) | (*x* $≤ #0 ∧ *y* $≤ #0))
⟨*proof*⟩

lemma *zmult-less-0-iff*:
(*x* $∗ *y* $< #0) ⟷ (#0 $< *x* ∧ *y* $< #0 | *x* $< #0 ∧ #0 $< *y*)
⟨*proof*⟩

lemma *zmult-le-0-iff*:
(*x* $∗ *y* $≤ #0) ⟷ (#0 $≤ *x* ∧ *y* $≤ #0 | *x* $≤ #0 ∧ #0 $≤ *y*)
⟨*proof*⟩

lemma *posDivAlg-type* [*rule-format*]:
⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ *posDivAlg*(⟨*a*,*b*⟩) ∈ *int* ∗ *int*
⟨*proof*⟩

lemma *posDivAlg-correct* [*rule-format*]:
⟦*a* ∈ *int*; *b* ∈ *int*⟧
⟹ #0 $≤ *a* ⟶ #0 $< *b* ⟶ *quorem* (⟨*a*,*b*⟩, *posDivAlg*(⟨*a*,*b*⟩))
⟨*proof*⟩

33.4 Correctness of negDivAlg, the division algorithm for a<0 and b>0

lemma *negDivAlg-termination*:
$[\![\#0 \ \$< b;\ a \ \$+ \ b \ \$< \#0]\!]$
$\Longrightarrow$ *nat-of*($\$- \ a \ \$- \ \#2 \ \$* \ b$) $<$ *nat-of*($\$- \ a \ \$- \ b$)
⟨*proof*⟩

lemmas *negDivAlg-unfold* = *def-wfrec* [*OF negDivAlg-def wf-measure*]

lemma *negDivAlg-eqn*:
$[\![\#0 \ \$< b;\ a \in int;\ b \in int]\!] \Longrightarrow$
negDivAlg(⟨*a*,*b*⟩) =
(*if* $\#0 \ \$\leq a\$+b$ *then* $<\#-1,a\$+b>$
else adjust(*b*, *negDivAlg* ($<a, \#2\$*b>$)))
⟨*proof*⟩

lemma *negDivAlg-induct-lemma* [*rule-format*]:
assumes *prem*:
$\bigwedge a\ b.\ [\![a \in int;\ b \in int;$
$\neg\ (\#0 \ \$\leq a \ \$+ \ b \mid b \ \$\leq \#0) \longrightarrow P(<a, \#2 \ \$* \ b>)]\!]$
$\Longrightarrow P(\langle a,b\rangle)$
shows $\langle u,v\rangle \in int*int \Longrightarrow P(\langle u,v\rangle)$
⟨*proof*⟩

lemma *negDivAlg-induct* [*consumes 2*]:
assumes *u-int*: $u \in int$
and *v-int*: $v \in int$
and *ih*: $\bigwedge a\ b.\ [\![a \in int;\ b \in int;$
$\neg\ (\#0 \ \$\leq a \ \$+ \ b \mid b \ \$\leq \#0) \longrightarrow P(a, \#2 \ \$* \ b)]\!]$
$\Longrightarrow P(a,b)$
shows $P(u,v)$
⟨*proof*⟩

lemma *negDivAlg-type*:
$[\![a \in int;\ b \in int]\!] \Longrightarrow$ *negDivAlg*(⟨*a*,*b*⟩) $\in int * int$
⟨*proof*⟩

lemma *negDivAlg-correct* [*rule-format*]:
$[\![a \in int;\ b \in int]\!]$
$\Longrightarrow a \ \$< \#0 \longrightarrow \#0 \ \$< b \longrightarrow$ *quorem* (⟨*a*,*b*⟩, *negDivAlg*(⟨*a*,*b*⟩))
⟨*proof*⟩

33.5 Existence shown by proving the division algorithm to be correct

lemma *quorem-0*: $[\![b \neq \#0;\ \ b \in int]\!] \Longrightarrow quorem\ (<\#0,b>,\ <\#0,\#0>)$
⟨*proof*⟩

lemma *posDivAlg-zero-divisor*: $posDivAlg(<a,\#0>) = <\#0,a>$
⟨*proof*⟩

lemma *posDivAlg-0* [*simp*]: $posDivAlg\ (<\#0,b>) = <\#0,\#0>$
⟨*proof*⟩

lemma *linear-arith-lemma*: $\neg\ (\#0\ \$\leq \#-1\ \$+\ b) \Longrightarrow (b\ \$\leq \#0)$
⟨*proof*⟩

lemma *negDivAlg-minus1* [*simp*]: $negDivAlg\ (<\#-1,b>) = <\#-1,\ b\$-\#1>$
⟨*proof*⟩

lemma *negateSnd-eq* [*simp*]: $negateSnd\ (\langle q,r\rangle) = <q,\ \$-r>$
⟨*proof*⟩

lemma *negateSnd-type*: $qr \in int * int \Longrightarrow negateSnd\ (qr) \in int * int$
⟨*proof*⟩

lemma *quorem-neg*:
$[\![quorem\ (<\$-a,\$-b>,\ qr);\ \ a \in int;\ \ b \in int;\ \ qr \in int * int]\!]$
$\Longrightarrow quorem\ (\langle a,b\rangle,\ negateSnd(qr))$
⟨*proof*⟩

lemma *divAlg-correct*:
$[\![b \neq \#0;\ \ a \in int;\ \ b \in int]\!] \Longrightarrow quorem\ (\langle a,b\rangle,\ divAlg(\langle a,b\rangle))$
⟨*proof*⟩

lemma *divAlg-type*: $[\![a \in int;\ \ b \in int]\!] \Longrightarrow divAlg(\langle a,b\rangle) \in int * int$
⟨*proof*⟩

lemma *zdiv-intify1* [*simp*]: $intify(x)\ zdiv\ y = x\ zdiv\ y$
⟨*proof*⟩

lemma *zdiv-intify2* [*simp*]: $x\ zdiv\ intify(y) = x\ zdiv\ y$
⟨*proof*⟩

lemma *zdiv-type* [*iff*, *TC*]: $z\ zdiv\ w \in int$
⟨*proof*⟩

lemma *zmod-intify1* [*simp*]: *intify*(*x*) *zmod y* = *x zmod y*
⟨*proof*⟩

lemma *zmod-intify2* [*simp*]: *x zmod intify*(*y*) = *x zmod y*
⟨*proof*⟩

lemma *zmod-type* [*iff*, *TC*]: *z zmod w* ∈ *int*
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-ZDIV*: *a zdiv* #*0* = #*0*
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-ZMOD*: *a zmod* #*0* = *intify*(*a*)
⟨*proof*⟩

lemma *raw-zmod-zdiv-equality*:
⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ *a* = *b* $∗ (*a zdiv b*) $+ (*a zmod b*)
⟨*proof*⟩

lemma *zmod-zdiv-equality*: *intify*(*a*) = *b* $∗ (*a zdiv b*) $+ (*a zmod b*)
⟨*proof*⟩

lemma *pos-mod*: #*0* $< *b* ⟹ #*0* $≤ *a zmod b* ∧ *a zmod b* $< *b*
⟨*proof*⟩

lemmas *pos-mod-sign* = *pos-mod* [*THEN conjunct1*]
and *pos-mod-bound* = *pos-mod* [*THEN conjunct2*]

lemma *neg-mod*: *b* $< #*0* ⟹ *a zmod b* $≤ #*0* ∧ *b* $< *a zmod b*
⟨*proof*⟩

lemmas *neg-mod-sign* = *neg-mod* [*THEN conjunct1*]
and *neg-mod-bound* = *neg-mod* [*THEN conjunct2*]

lemma *quorem-div-mod*:
⟦*b* ≠ #*0*; *a* ∈ *int*; *b* ∈ *int*⟧
⟹ *quorem* (⟨*a*,*b*⟩, <*a zdiv b*, *a zmod b*>)
⟨*proof*⟩

lemma *quorem-div*:
$[\![quorem(\langle a,b\rangle,\langle q,r\rangle);\ b \neq \#0;\ a \in int;\ b \in int;\ q \in int]\!]$
$\Longrightarrow a\ zdiv\ b = q$
⟨*proof*⟩

lemma *quorem-mod*:
$[\![quorem(\langle a,b\rangle,\langle q,r\rangle);\ b \neq \#0;\ a \in int;\ b \in int;\ q \in int;\ r \in int]\!]$
$\Longrightarrow a\ zmod\ b = r$
⟨*proof*⟩

lemma *zdiv-pos-pos-trivial-raw*:
$[\![a \in int;\ b \in int;\ \#0\ \$\leq a;\ a\ \$< b]\!] \Longrightarrow a\ zdiv\ b = \#0$
⟨*proof*⟩

lemma *zdiv-pos-pos-trivial*: $[\![\#0\ \$\leq a;\ a\ \$< b]\!] \Longrightarrow a\ zdiv\ b = \#0$
⟨*proof*⟩

lemma *zdiv-neg-neg-trivial-raw*:
$[\![a \in int;\ b \in int;\ a\ \$\leq \#0;\ b\ \$< a]\!] \Longrightarrow a\ zdiv\ b = \#0$
⟨*proof*⟩

lemma *zdiv-neg-neg-trivial*: $[\![a\ \$\leq \#0;\ b\ \$< a]\!] \Longrightarrow a\ zdiv\ b = \#0$
⟨*proof*⟩

lemma *zadd-le-0-lemma*: $[\![a\$+b\ \$\leq \#0;\ \#0\ \$< a;\ \#0\ \$< b]\!] \Longrightarrow False$
⟨*proof*⟩

lemma *zdiv-pos-neg-trivial-raw*:
$[\![a \in int;\ b \in int;\ \#0\ \$< a;\ a\$+b\ \$\leq \#0]\!] \Longrightarrow a\ zdiv\ b = \#-1$
⟨*proof*⟩

lemma *zdiv-pos-neg-trivial*: $[\![\#0\ \$< a;\ a\$+b\ \$\leq \#0]\!] \Longrightarrow a\ zdiv\ b = \#-1$
⟨*proof*⟩

lemma *zmod-pos-pos-trivial-raw*:
$[\![a \in int;\ b \in int;\ \#0\ \$\leq a;\ a\ \$< b]\!] \Longrightarrow a\ zmod\ b = a$
⟨*proof*⟩

lemma *zmod-pos-pos-trivial*: $[\![\#0\ \$\leq a;\ a\ \$< b]\!] \Longrightarrow a\ zmod\ b = intify(a)$
⟨*proof*⟩

lemma *zmod-neg-neg-trivial-raw*:
$[\![a \in int;\ b \in int;\ a\ \$\leq \#0;\ b\ \$< a]\!] \Longrightarrow a\ zmod\ b = a$
⟨*proof*⟩

lemma *zmod-neg-neg-trivial*: $[\![a\ \$\leq \#0;\ b\ \$< a]\!] \Longrightarrow a\ zmod\ b = intify(a)$

⟨*proof*⟩

lemma *zmod-pos-neg-trivial-raw*:
⟦$a \in int$; $b \in int$; $\#0\ \$< a$; $a\$+b\ \$\leq \#0$⟧ ⟹ $a\ zmod\ b = a\$+b$
⟨*proof*⟩

lemma *zmod-pos-neg-trivial*: ⟦$\#0\ \$< a$; $a\$+b\ \$\leq \#0$⟧ ⟹ $a\ zmod\ b = a\$+b$
⟨*proof*⟩

lemma *zdiv-zminus-zminus-raw*:
⟦$a \in int$; $b \in int$⟧ ⟹ $(\$-a)\ zdiv\ (\$-b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zdiv-zminus-zminus* [*simp*]: $(\$-a)\ zdiv\ (\$-b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zmod-zminus-zminus-raw*:
⟦$a \in int$; $b \in int$⟧ ⟹ $(\$-a)\ zmod\ (\$-b) = \$-\ (a\ zmod\ b)$
⟨*proof*⟩

lemma *zmod-zminus-zminus* [*simp*]: $(\$-a)\ zmod\ (\$-b) = \$-\ (a\ zmod\ b)$
⟨*proof*⟩

33.6 division of a number by itself

lemma *self-quotient-aux1*: ⟦$\#0\ \$< a$; $a = r\ \$+\ a\$*q$; $r\ \$< a$⟧ ⟹ $\#1\ \$\leq q$
⟨*proof*⟩

lemma *self-quotient-aux2*: ⟦$\#0\ \$< a$; $a = r\ \$+\ a\$*q$; $\#0\ \$\leq r$⟧ ⟹ $q\ \$\leq \#1$
⟨*proof*⟩

lemma *self-quotient*:
⟦$quorem(\langle a,a\rangle,\langle q,r\rangle)$; $a \in int$; $q \in int$; $a \neq \#0$⟧ ⟹ $q = \#1$
⟨*proof*⟩

lemma *self-remainder*:
⟦$quorem(\langle a,a\rangle,\langle q,r\rangle)$; $a \in int$; $q \in int$; $r \in int$; $a \neq \#0$⟧ ⟹ $r = \#0$
⟨*proof*⟩

lemma *zdiv-self-raw*: ⟦$a \neq \#0$; $a \in int$⟧ ⟹ $a\ zdiv\ a = \#1$
⟨*proof*⟩

lemma *zdiv-self* [*simp*]: $intify(a) \neq \#0 \Longrightarrow a\ zdiv\ a = \#1$

⟨*proof*⟩

lemma *zmod-self-raw*: $a \in int \Longrightarrow a\ zmod\ a = \#0$
⟨*proof*⟩

lemma *zmod-self* [*simp*]: $a\ zmod\ a = \#0$
⟨*proof*⟩

33.7 Computation of division and remainder

lemma *zdiv-zero* [*simp*]: $\#0\ zdiv\ b = \#0$
⟨*proof*⟩

lemma *zdiv-eq-minus1*: $\#0\ \$< b \Longrightarrow \#-1\ zdiv\ b = \#-1$
⟨*proof*⟩

lemma *zmod-zero* [*simp*]: $\#0\ zmod\ b = \#0$
⟨*proof*⟩

lemma *zdiv-minus1*: $\#0\ \$< b \Longrightarrow \#-1\ zdiv\ b = \#-1$
⟨*proof*⟩

lemma *zmod-minus1*: $\#0\ \$< b \Longrightarrow \#-1\ zmod\ b = b\ \$- \#1$
⟨*proof*⟩

lemma *zdiv-pos-pos*: $[\![\#0\ \$< a;\ \#0\ \$\leq b]\!]$
$\Longrightarrow a\ zdiv\ b = fst\ (posDivAlg(<intify(a),\ intify(b)>))$
⟨*proof*⟩

lemma *zmod-pos-pos*:
$[\![\#0\ \$< a;\ \#0\ \$\leq b]\!]$
$\Longrightarrow a\ zmod\ b = snd\ (posDivAlg(<intify(a),\ intify(b)>))$
⟨*proof*⟩

lemma *zdiv-neg-pos*:
$[\![a\ \$< \#0;\ \#0\ \$< b]\!]$
$\Longrightarrow a\ zdiv\ b = fst\ (negDivAlg(<intify(a),\ intify(b)>))$
⟨*proof*⟩

lemma *zmod-neg-pos*:
$[\![a\ \$< \#0;\ \#0\ \$< b]\!]$
$\Longrightarrow a\ zmod\ b = snd\ (negDivAlg(<intify(a),\ intify(b)>))$
⟨*proof*⟩

lemma *zdiv-pos-neg*:
$[\![\#0\ \$<\ a;\ \ b\ \$<\ \#0]\!]$
$\Longrightarrow a\ zdiv\ b = fst\ (negateSnd(negDivAlg\ (<\$-a,\ \$-b>)))$
⟨*proof*⟩

lemma *zmod-pos-neg*:
$[\![\#0\ \$<\ a;\ \ b\ \$<\ \#0]\!]$
$\Longrightarrow a\ zmod\ b = snd\ (negateSnd(negDivAlg\ (<\$-a,\ \$-b>)))$
⟨*proof*⟩

lemma *zdiv-neg-neg*:
$[\![a\ \$<\ \#0;\ \ b\ \$\leq\ \#0]\!]$
$\Longrightarrow a\ zdiv\ b = fst\ (negateSnd(posDivAlg(<\$-a,\ \$-b>)))$
⟨*proof*⟩

lemma *zmod-neg-neg*:
$[\![a\ \$<\ \#0;\ \ b\ \$\leq\ \#0]\!]$
$\Longrightarrow a\ zmod\ b = snd\ (negateSnd(posDivAlg(<\$-a,\ \$-b>)))$
⟨*proof*⟩

declare *zdiv-pos-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-neg-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-pos-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zdiv-neg-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-pos-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-neg-pos* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-pos-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *zmod-neg-neg* [*of integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *posDivAlg-eqn* [*of concl*: *integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*
declare *negDivAlg-eqn* [*of concl*: *integ-of* (*v*) *integ-of* (*w*), *simp*] **for** *v w*

lemma *zmod-1* [*simp*]: $a\ zmod\ \#1 = \#0$
⟨*proof*⟩

lemma *zdiv-1* [*simp*]: $a\ zdiv\ \#1 = intify(a)$
⟨*proof*⟩

lemma *zmod-minus1-right* [*simp*]: $a\ zmod\ \#-1 = \#0$
⟨*proof*⟩

lemma *zdiv-minus1-right-raw*: $a \in int \Longrightarrow a\ zdiv\ \#-1 = \$-a$
⟨*proof*⟩

lemma *zdiv-minus1-right*: $a \; zdiv \; \#{-}1 = \${-}a$
⟨*proof*⟩
declare *zdiv-minus1-right* [*simp*]

33.8 Monotonicity in the first argument (divisor)

lemma *zdiv-mono1*: $[\![a \; \$\leq \; a'; \;\; \#0 \; \$< \; b]\!] \Longrightarrow a \; zdiv \; b \; \$\leq \; a' \; zdiv \; b$
⟨*proof*⟩

lemma *zdiv-mono1-neg*: $[\![a \; \$\leq \; a'; \;\; b \; \$< \; \#0]\!] \Longrightarrow a' \; zdiv \; b \; \$\leq \; a \; zdiv \; b$
⟨*proof*⟩

33.9 Monotonicity in the second argument (dividend)

lemma *q-pos-lemma*:
$[\![\#0 \; \$\leq \; b'\${*}q' \; \$+ \; r'; \; r' \; \$< \; b'; \;\; \#0 \; \$< \; b']\!] \Longrightarrow \#0 \; \$\leq \; q'$
⟨*proof*⟩

lemma *zdiv-mono2-lemma*:
$[\![b\${*}q \; \$+ \; r = b'\${*}q' \; \$+ \; r'; \;\; \#0 \; \$\leq \; b'\${*}q' \; \$+ \; r'; \;\; r' \; \$< \; b'; \;\; \#0 \; \$\leq \; r; \;\; \#0 \; \$< \; b'; \;\; b' \; \$\leq \; b]\!] \Longrightarrow q \; \$\leq \; q'$
⟨*proof*⟩

lemma *zdiv-mono2-raw*:
$[\![\#0 \; \$\leq \; a; \;\; \#0 \; \$< \; b'; \;\; b' \; \$\leq \; b; \;\; a \in int]\!] \Longrightarrow a \; zdiv \; b \; \$\leq \; a \; zdiv \; b'$
⟨*proof*⟩

lemma *zdiv-mono2*:
$[\![\#0 \; \$\leq \; a; \;\; \#0 \; \$< \; b'; \;\; b' \; \$\leq \; b]\!] \Longrightarrow a \; zdiv \; b \; \$\leq \; a \; zdiv \; b'$
⟨*proof*⟩

lemma *q-neg-lemma*:
$[\![b'\${*}q' \; \$+ \; r' \; \$< \; \#0; \;\; \#0 \; \$\leq \; r'; \;\; \#0 \; \$< \; b']\!] \Longrightarrow q' \; \$< \; \#0$
⟨*proof*⟩

lemma *zdiv-mono2-neg-lemma*:
$[\![b\${*}q \; \$+ \; r = b'\${*}q' \; \$+ \; r'; \;\; b'\${*}q' \; \$+ \; r' \; \$< \; \#0; \;\; r \; \$< \; b; \;\; \#0 \; \$\leq \; r'; \;\; \#0 \; \$< \; b'; \;\; b' \; \$\leq \; b]\!] \Longrightarrow q' \; \$\leq \; q$
⟨*proof*⟩

lemma *zdiv-mono2-neg-raw*:
$[\![a \; \$< \; \#0; \;\; \#0 \; \$< \; b'; \;\; b' \; \$\leq \; b; \;\; a \in int]\!]$

$\Longrightarrow$ *a zdiv b′ $≤ a zdiv b*
⟨*proof*⟩

lemma *zdiv-mono2-neg*: ⟦*a $< #0*; *#0 $< b′*; *b′ $≤ b*⟧
$\Longrightarrow$ *a zdiv b′ $≤ a zdiv b*
⟨*proof*⟩

33.10 More algebraic laws for zdiv and zmod

lemma *zmult1-lemma*:
⟦*quorem*(⟨*b*,*c*⟩, ⟨*q*,*r*⟩); *c* ∈ *int*; *c* ≠ *#0*⟧
$\Longrightarrow$ *quorem* (<*a$*b*, *c*>, <*a$*q $+ (a$*r*) *zdiv c*, (*a$*r*) *zmod c*>)
⟨*proof*⟩

lemma *zdiv-zmult1-eq-raw*:
⟦*b* ∈ *int*; *c* ∈ *int*⟧
$\Longrightarrow$ (*a$*b*) *zdiv c* = *a$*(b zdiv c) $+ a$*(b zmod c) zdiv c*
⟨*proof*⟩

lemma *zdiv-zmult1-eq*: (*a$*b*) *zdiv c* = *a$*(b zdiv c) $+ a$*(b zmod c) zdiv c*
⟨*proof*⟩

lemma *zmod-zmult1-eq-raw*:
⟦*b* ∈ *int*; *c* ∈ *int*⟧ $\Longrightarrow$ (*a$*b*) *zmod c* = *a$*(b zmod c) zmod c*
⟨*proof*⟩

lemma *zmod-zmult1-eq*: (*a$*b*) *zmod c* = *a$*(b zmod c) zmod c*
⟨*proof*⟩

lemma *zmod-zmult1-eq′*: (*a$*b*) *zmod c* = ((*a zmod c*) $* *b*) *zmod c*
⟨*proof*⟩

lemma *zmod-zmult-distrib*: (*a$*b*) *zmod c* = ((*a zmod c*) $* (*b zmod c*)) *zmod c*
⟨*proof*⟩

lemma *zdiv-zmult-self1* [*simp*]: *intify*(*b*) ≠ *#0* $\Longrightarrow$ (*a$*b*) *zdiv b* = *intify*(*a*)
⟨*proof*⟩

lemma *zdiv-zmult-self2* [*simp*]: *intify*(*b*) ≠ *#0* $\Longrightarrow$ (*b$*a*) *zdiv b* = *intify*(*a*)
⟨*proof*⟩

lemma *zmod-zmult-self1* [*simp*]: (*a$*b*) *zmod b* = *#0*
⟨*proof*⟩

lemma *zmod-zmult-self2* [*simp*]: (*b$*a*) *zmod b* = *#0*
⟨*proof*⟩

lemma *zadd1-lemma*:
⟦*quorem*(⟨*a*,*c*⟩, ⟨*aq*,*ar*⟩); *quorem*(⟨*b*,*c*⟩, ⟨*bq*,*br*⟩);
c ∈ *int*; *c* ≠ #0⟧
⟹ *quorem* (<*a*$+*b*, *c*>, <*aq* $+ *bq* $+ (*ar*$+*br*) *zdiv c*, (*ar*$+*br*) *zmod c*>)
⟨*proof*⟩

lemma *zdiv-zadd1-eq-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*; *c* ∈ *int*⟧ ⟹
(*a*$+*b*) *zdiv c* = *a zdiv c* $+ *b zdiv c* $+ ((*a zmod c* $+ *b zmod c*) *zdiv c*)
⟨*proof*⟩

lemma *zdiv-zadd1-eq*:
(*a*$+*b*) *zdiv c* = *a zdiv c* $+ *b zdiv c* $+ ((*a zmod c* $+ *b zmod c*) *zdiv c*)
⟨*proof*⟩

lemma *zmod-zadd1-eq-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*; *c* ∈ *int*⟧
⟹ (*a*$+*b*) *zmod c* = (*a zmod c* $+ *b zmod c*) *zmod c*
⟨*proof*⟩

lemma *zmod-zadd1-eq*: (*a*$+*b*) *zmod c* = (*a zmod c* $+ *b zmod c*) *zmod c*
⟨*proof*⟩

lemma *zmod-div-trivial-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ (*a zmod b*) *zdiv b* = #0
⟨*proof*⟩

lemma *zmod-div-trivial* [*simp*]: (*a zmod b*) *zdiv b* = #0
⟨*proof*⟩

lemma *zmod-mod-trivial-raw*:
⟦*a* ∈ *int*; *b* ∈ *int*⟧ ⟹ (*a zmod b*) *zmod b* = *a zmod b*
⟨*proof*⟩

lemma *zmod-mod-trivial* [*simp*]: (*a zmod b*) *zmod b* = *a zmod b*
⟨*proof*⟩

lemma *zmod-zadd-left-eq*: (*a*$+*b*) *zmod c* = ((*a zmod c*) $+ *b*) *zmod c*
⟨*proof*⟩

lemma *zmod-zadd-right-eq*: (*a*$+*b*) *zmod c* = (*a* $+ (*b zmod c*)) *zmod c*
⟨*proof*⟩

lemma *zdiv-zadd-self1* [*simp*]:
intify(*a*) ≠ #0 ⟹ (*a*$+*b*) *zdiv a* = *b zdiv a* $+ #1
⟨*proof*⟩

lemma *zdiv-zadd-self2* [*simp*]:
intify(*a*) ≠ #0 ⟹ (*b*$+*a*) *zdiv a* = *b zdiv a* $+ #1
⟨*proof*⟩

lemma *zmod-zadd-self1* [*simp*]: (*a*$+*b*) *zmod a* = *b zmod a*
⟨*proof*⟩

lemma *zmod-zadd-self2* [*simp*]: (*b*$+*a*) *zmod a* = *b zmod a*
⟨*proof*⟩

33.11 proving a zdiv (b*c) = (a zdiv b) zdiv c

lemma *zdiv-zmult2-aux1*:
⟦#0 $< *c*; *b* $< *r*; *r* $≤ #0⟧ ⟹ *b*$∗*c* $< *b*$∗(*q zmod c*) $+ *r*
⟨*proof*⟩

lemma *zdiv-zmult2-aux2*:
⟦#0 $< *c*; *b* $< *r*; *r* $≤ #0⟧ ⟹ *b* $∗ (*q zmod c*) $+ *r* $≤ #0
⟨*proof*⟩

lemma *zdiv-zmult2-aux3*:
⟦#0 $< *c*; #0 $≤ *r*; *r* $< *b*⟧ ⟹ #0 $≤ *b* $∗ (*q zmod c*) $+ *r*
⟨*proof*⟩

lemma *zdiv-zmult2-aux4*:
⟦#0 $< *c*; #0 $≤ *r*; *r* $< *b*⟧ ⟹ *b* $∗ (*q zmod c*) $+ *r* $< *b* $∗ *c*
⟨*proof*⟩

lemma *zdiv-zmult2-lemma*:
⟦*quorem* (⟨*a*,*b*⟩, ⟨*q*,*r*⟩); *a* ∈ *int*; *b* ∈ *int*; *b* ≠ #0; #0 $< *c*⟧
⟹ *quorem* (<*a*,*b*$∗*c*>, <*q zdiv c*, *b*$∗(*q zmod c*) $+ *r*>)
⟨*proof*⟩

lemma *zdiv-zmult2-eq-raw*:
⟦#0 $< *c*; *a* ∈ *int*; *b* ∈ *int*⟧ ⟹ *a zdiv* (*b*$∗*c*) = (*a zdiv b*) *zdiv c*
⟨*proof*⟩

lemma *zdiv-zmult2-eq*: #0 $< *c* ⟹ *a zdiv* (*b*$∗*c*) = (*a zdiv b*) *zdiv c*
⟨*proof*⟩

lemma *zmod-zmult2-eq-raw*:
⟦#0 $< *c*; *a* ∈ *int*; *b* ∈ *int*⟧
⟹ *a zmod* (*b*$∗*c*) = *b*$∗(*a zdiv b zmod c*) $+ *a zmod b*
⟨*proof*⟩

lemma *zmod-zmult2-eq*:
#0 $< *c* ⟹ *a zmod* (*b*$∗*c*) = *b*$∗(*a zdiv b zmod c*) $+ *a zmod b*
⟨*proof*⟩

33.12 Cancellation of common factors in "zdiv"

lemma *zdiv-zmult-zmult1-aux1*:
$[\![\#0 \ \$< b;\ \ intify(c) \neq \#0]\!] \Longrightarrow (c\$*a)\ zdiv\ (c\$*b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1-aux2*:
$[\![b \ \$< \#0;\ \ intify(c) \neq \#0]\!] \Longrightarrow (c\$*a)\ zdiv\ (c\$*b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1-raw*:
$[\![intify(c) \neq \#0;\ b \in int]\!] \Longrightarrow (c\$*a)\ zdiv\ (c\$*b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1*: $intify(c) \neq \#0 \Longrightarrow (c\$*a)\ zdiv\ (c\$*b) = a\ zdiv\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult2*: $intify(c) \neq \#0 \Longrightarrow (a\$*c)\ zdiv\ (b\$*c) = a\ zdiv\ b$
⟨*proof*⟩

33.13 Distribution of factors over "zmod"

lemma *zmod-zmult-zmult1-aux1*:
$[\![\#0 \ \$< b;\ \ intify(c) \neq \#0]\!]$
$\Longrightarrow (c\$*a)\ zmod\ (c\$*b) = c\ \$*\ (a\ zmod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult1-aux2*:
$[\![b \ \$< \#0;\ \ intify(c) \neq \#0]\!]$
$\Longrightarrow (c\$*a)\ zmod\ (c\$*b) = c\ \$*\ (a\ zmod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult1-raw*:
$[\![b \in int;\ c \in int]\!] \Longrightarrow (c\$*a)\ zmod\ (c\$*b) = c\ \$*\ (a\ zmod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult1*: $(c\$*a)\ zmod\ (c\$*b) = c\ \$*\ (a\ zmod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult2*: $(a\$*c)\ zmod\ (b\$*c) = (a\ zmod\ b)\ \$*\ c$
⟨*proof*⟩

lemma *zdiv-neg-pos-less0*: $[\![a \ \$< \#0;\ \ \#0 \ \$< b]\!] \Longrightarrow a\ zdiv\ b \ \$< \#0$
⟨*proof*⟩

lemma *zdiv-nonneg-neg-le0*: $[\![\#0 \ \$\leq a;\ \ b \ \$< \#0]\!] \Longrightarrow a\ zdiv\ b \ \$\leq \#0$
⟨*proof*⟩

lemma *pos-imp-zdiv-nonneg-iff*: $\#0 \ \$< \ b \Longrightarrow (\#0 \ \$\leq \ a \ zdiv \ b) \longleftrightarrow (\#0 \ \$\leq \ a)$
⟨*proof*⟩

lemma *neg-imp-zdiv-nonneg-iff*: $b \ \$< \ \#0 \Longrightarrow (\#0 \ \$\leq \ a \ zdiv \ b) \longleftrightarrow (a \ \$\leq \ \#0)$
⟨*proof*⟩

lemma *pos-imp-zdiv-neg-iff*: $\#0 \ \$< \ b \Longrightarrow (a \ zdiv \ b \ \$< \ \#0) \longleftrightarrow (a \ \$< \ \#0)$
⟨*proof*⟩

lemma *neg-imp-zdiv-neg-iff*: $b \ \$< \ \#0 \Longrightarrow (a \ zdiv \ b \ \$< \ \#0) \longleftrightarrow (\#0 \ \$< \ a)$
⟨*proof*⟩

end

34 Cardinal Arithmetic Without the Axiom of Choice

theory *CardinalArith* **imports** *Cardinal OrderArith ArithSimp Finite* **begin**

definition
InfCard :: $i \Rightarrow o$ **where**
$InfCard(i) \equiv Card(i) \wedge nat \leq i$

definition
cmult :: $[i,i] \Rightarrow i$ (**infixl** ‹⊗› 70) **where**
$i \otimes j \equiv |i * j|$

definition
cadd :: $[i,i] \Rightarrow i$ (**infixl** ‹⊕› 65) **where**
$i \oplus j \equiv |i + j|$

definition
csquare-rel :: $i \Rightarrow i$ **where**
csquare-rel$(K) \equiv$
rvimage$(K*K,$
lam $\langle x,y\rangle : K*K. \ <x \cup y, \ x, \ y>,$
rmult$(K, Memrel(K), \ K*K, \ rmult(K, Memrel(K), \ K, Memrel(K))))$

definition
jump-cardinal :: $i \Rightarrow i$ **where**
— This definition is more complex than Kunen's but it more easily proved to be a cardinal
jump-cardinal$(K) \equiv$
$\bigcup X \in Pow(K). \ \{z. \ r \in Pow(K*K), \ well\text{-}ord(X,r) \wedge z = ordertype(X,r)\}$

definition
csucc :: $i \Rightarrow i$ **where**
— needed because *jump-cardinal*(K) might not be the successor of K
csucc(K) $\equiv \mu\ L.\ Card(L) \wedge K<L$

lemma *Card-Union* [*simp*,*intro*,*TC*]:
assumes A: $\bigwedge x.\ x \in A \Longrightarrow Card(x)$ **shows** $Card(\bigcup(A))$
⟨*proof*⟩

lemma *Card-UN*: $(\bigwedge x.\ x \in A \Longrightarrow Card(K(x))) \Longrightarrow Card(\bigcup x \in A.\ K(x))$
⟨*proof*⟩

lemma *Card-OUN* [*simp*,*intro*,*TC*]:
$(\bigwedge x.\ x \in A \Longrightarrow Card(K(x))) \Longrightarrow Card(\bigcup x<A.\ K(x))$
⟨*proof*⟩

lemma *in-Card-imp-lesspoll*: $[\![Card(K);\ b \in K]\!] \Longrightarrow b \prec K$
⟨*proof*⟩

34.1 Cardinal addition

Note: Could omit proving the algebraic laws for cardinal addition and multiplication. On finite cardinals these operations coincide with addition and multiplication of natural numbers; on infinite cardinals they coincide with union (maximum). Either way we get most laws for free.

34.1.1 Cardinal addition is commutative

lemma *sum-commute-eqpoll*: $A+B \approx B+A$
⟨*proof*⟩

lemma *cadd-commute*: $i \oplus j = j \oplus i$
⟨*proof*⟩

34.1.2 Cardinal addition is associative

lemma *sum-assoc-eqpoll*: $(A+B)+C \approx A+(B+C)$
⟨*proof*⟩

Unconditional version requires AC

lemma *well-ord-cadd-assoc*:
assumes i: *well-ord*(i,ri) **and** j: *well-ord*(j,rj) **and** k: *well-ord*(k,rk)
shows $(i \oplus j) \oplus k = i \oplus (j \oplus k)$
⟨*proof*⟩

34.1.3 0 is the identity for addition

lemma *sum-0-eqpoll*: $0+A \approx A$

⟨*proof*⟩

lemma *cadd-0* [*simp*]: $Card(K) \Longrightarrow 0 \oplus K = K$
⟨*proof*⟩

34.1.4 Addition by another cardinal

lemma *sum-lepoll-self*: $A \lesssim A+B$
⟨*proof*⟩

lemma *cadd-le-self*:
assumes K: $Card(K)$ **and** L: $Ord(L)$ **shows** $K \leq (K \oplus L)$
⟨*proof*⟩

34.1.5 Monotonicity of addition

lemma *sum-lepoll-mono*:
$[\![A \lesssim C;\ B \lesssim D]\!] \Longrightarrow A + B \lesssim C + D$
⟨*proof*⟩

lemma *cadd-le-mono*:
$[\![K' \leq K;\ L' \leq L]\!] \Longrightarrow (K' \oplus L') \leq (K \oplus L)$
⟨*proof*⟩

34.1.6 Addition of finite cardinals is "ordinary" addition

lemma *sum-succ-eqpoll*: $succ(A)+B \approx succ(A+B)$
⟨*proof*⟩

lemma *cadd-succ-lemma*:
assumes $Ord(m)$ $Ord(n)$ **shows** $succ(m) \oplus n = |succ(m \oplus n)|$
⟨*proof*⟩

lemma *nat-cadd-eq-add*:
assumes m: $m \in nat$ **and** [*simp*]: $n \in nat$ **shows**$m \oplus n = m \#+ n$
⟨*proof*⟩

34.2 Cardinal multiplication

34.2.1 Cardinal multiplication is commutative

lemma *prod-commute-eqpoll*: $A*B \approx B*A$
⟨*proof*⟩

lemma *cmult-commute*: $i \otimes j = j \otimes i$
⟨*proof*⟩

34.2.2 Cardinal multiplication is associative

lemma *prod-assoc-eqpoll*: $(A*B)*C \approx A*(B*C)$
⟨*proof*⟩

Unconditional version requires AC

lemma *well-ord-cmult-assoc*:
assumes *i*: *well-ord*(*i*,*ri*) **and** *j*: *well-ord*(*j*,*rj*) **and** *k*: *well-ord*(*k*,*rk*)
shows $(i \otimes j) \otimes k = i \otimes (j \otimes k)$
⟨*proof*⟩

34.2.3 Cardinal multiplication distributes over addition

lemma *sum-prod-distrib-eqpoll*: $(A+B)*C \approx (A*C)+(B*C)$
⟨*proof*⟩

lemma *well-ord-cadd-cmult-distrib*:
assumes *i*: *well-ord*(*i*,*ri*) **and** *j*: *well-ord*(*j*,*rj*) **and** *k*: *well-ord*(*k*,*rk*)
shows $(i \oplus j) \otimes k = (i \otimes k) \oplus (j \otimes k)$
⟨*proof*⟩

34.2.4 Multiplication by 0 yields 0

lemma *prod-0-eqpoll*: $0*A \approx 0$
⟨*proof*⟩

lemma *cmult-0* [*simp*]: $0 \otimes i = 0$
⟨*proof*⟩

34.2.5 1 is the identity for multiplication

lemma *prod-singleton-eqpoll*: $\{x\}*A \approx A$
⟨*proof*⟩

lemma *cmult-1* [*simp*]: *Card*(*K*) $\Longrightarrow 1 \otimes K = K$
⟨*proof*⟩

34.3 Some inequalities for multiplication

lemma *prod-square-lepoll*: $A \lesssim A*A$
⟨*proof*⟩

lemma *cmult-square-le*: *Card*(*K*) $\Longrightarrow K \le K \otimes K$
⟨*proof*⟩

34.3.1 Multiplication by a non-zero cardinal

lemma *prod-lepoll-self*: $b \in B \Longrightarrow A \lesssim A*B$
⟨*proof*⟩

lemma *cmult-le-self*:
$[\![Card(K);\ \ Ord(L);\ \ 0<L]\!] \Longrightarrow K \leq (K \otimes L)$
⟨*proof*⟩

34.3.2 Monotonicity of multiplication

lemma *prod-lepoll-mono*:
$[\![A \lesssim C;\ \ B \lesssim D]\!] \Longrightarrow A * B \ \lesssim\ C * D$
⟨*proof*⟩

lemma *cmult-le-mono*:
$[\![K' \leq K;\ \ L' \leq L]\!] \Longrightarrow (K' \otimes L') \leq (K \otimes L)$
⟨*proof*⟩

34.4 Multiplication of finite cardinals is "ordinary" multiplication

lemma *prod-succ-eqpoll*: $succ(A)*B \approx B + A*B$
⟨*proof*⟩

lemma *cmult-succ-lemma*:
$[\![Ord(m);\ \ Ord(n)]\!] \Longrightarrow succ(m) \otimes n = n \oplus (m \otimes n)$
⟨*proof*⟩

lemma *nat-cmult-eq-mult*: $[\![m \in nat;\ \ n \in nat]\!] \Longrightarrow m \otimes n = m\#*n$
⟨*proof*⟩

lemma *cmult-2*: $Card(n) \Longrightarrow 2 \otimes n = n \oplus n$
⟨*proof*⟩

lemma *sum-lepoll-prod*:
assumes C: $2 \lesssim C$ **shows** $B+B \lesssim C*B$
⟨*proof*⟩

lemma *lepoll-imp-sum-lepoll-prod*: $[\![A \lesssim B;\ 2 \lesssim A]\!] \Longrightarrow A+B \lesssim A*B$
⟨*proof*⟩

34.5 Infinite Cardinals are Limit Ordinals

lemma *nat-cons-lepoll*: $nat \lesssim A \Longrightarrow cons(u,A) \lesssim A$
⟨*proof*⟩

lemma *nat-cons-eqpoll*: $nat \lesssim A \Longrightarrow cons(u,A) \approx A$
⟨*proof*⟩

lemma *nat-succ-eqpoll*: $nat \subseteq A \Longrightarrow succ(A) \approx A$

⟨*proof*⟩

lemma *InfCard-nat*: *InfCard*(*nat*)
⟨*proof*⟩

lemma *InfCard-is-Card*: *InfCard*(*K*) ⟹ *Card*(*K*)
⟨*proof*⟩

lemma *InfCard-Un*:
⟦*InfCard*(*K*); *Card*(*L*)⟧ ⟹ *InfCard*(*K* ∪ *L*)
⟨*proof*⟩

lemma *InfCard-is-Limit*: *InfCard*(*K*) ⟹ *Limit*(*K*)
⟨*proof*⟩

lemma *ordermap-eqpoll-pred*:
⟦*well-ord*(*A*,*r*); *x* ∈ *A*⟧ ⟹ *ordermap*(*A*,*r*)‘*x* ≈ *Order.pred*(*A*,*x*,*r*)
⟨*proof*⟩

34.5.1 Establishing the well-ordering

lemma *well-ord-csquare*:
assumes *K*: *Ord*(*K*) **shows** *well-ord*(*K*∗*K*, *csquare-rel*(*K*))
⟨*proof*⟩

34.5.2 Characterising initial segments of the well-ordering

lemma *csquareD*:
⟦<⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*); *x*<*K*; *y*<*K*; *z*<*K*⟧ ⟹ *x* ≤ *z* ∧ *y* ≤ *z*
⟨*proof*⟩

lemma *pred-csquare-subset*:
z<*K* ⟹ *Order.pred*(*K*∗*K*, ⟨*z*,*z*⟩, *csquare-rel*(*K*)) ⊆ *succ*(*z*)∗*succ*(*z*)
⟨*proof*⟩

lemma *csquare-ltI*:
⟦*x*<*z*; *y*<*z*; *z*<*K*⟧ ⟹ <⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*)
⟨*proof*⟩

lemma *csquare-or-eqI*:
⟦*x* ≤ *z*; *y* ≤ *z*; *z*<*K*⟧ ⟹ <⟨*x*,*y*⟩, ⟨*z*,*z*⟩> ∈ *csquare-rel*(*K*) | *x*=*z* ∧ *y*=*z*
⟨*proof*⟩

34.5.3 The cardinality of initial segments

lemma *ordermap-z-lt*:
$[\![Limit(K);\ x{<}K;\ y{<}K;\ z{=}succ(x \cup y)]\!] \Longrightarrow$
$ordermap(K{*}K,\ csquare\text{-}rel(K))\ `\ \langle x,y\rangle <$
$ordermap(K{*}K,\ csquare\text{-}rel(K))\ `\ \langle z,z\rangle$
⟨*proof*⟩

Kunen: "each $\langle x,\ y\rangle \in K \times K$ has no more than $z \times z$ predecessors..." (page 29)

lemma *ordermap-csquare-le*:
assumes K: $Limit(K)$ **and** x: $x{<}K$ **and** y: $y{<}K$
defines $z \equiv succ(x \cup y)$
shows $|ordermap(K \times K,\ csquare\text{-}rel(K))\ `\ \langle x,y\rangle| \le |succ(z)| \otimes |succ(z)|$
⟨*proof*⟩

Kunen: "... so the order type is $\le$ K"

lemma *ordertype-csquare-le*:
assumes *IK*: $InfCard(K)$ **and** *eq*: $\bigwedge y.\ y{\in}K \Longrightarrow InfCard(y) \Longrightarrow y \otimes y = y$
shows $ordertype(K{*}K,\ csquare\text{-}rel(K)) \le K$
⟨*proof*⟩

lemma *InfCard-csquare-eq*:
assumes *IK*: $InfCard(K)$ **shows** $K \otimes K = K$
⟨*proof*⟩

lemma *well-ord-InfCard-square-eq*:
assumes r: $well\text{-}ord(A,r)$ **and** I: $InfCard(|A|)$ **shows** $A \times A \approx A$
⟨*proof*⟩

lemma *InfCard-square-eqpoll*: $InfCard(K) \Longrightarrow K \times K \approx K$
⟨*proof*⟩

lemma *Inf-Card-is-InfCard*: $[\![Card(i);\ \neg\ Finite(i)]\!] \Longrightarrow InfCard(i)$
⟨*proof*⟩

34.5.4 Toward's Kunen's Corollary 10.13 (1)

lemma *InfCard-le-cmult-eq*: $[\![InfCard(K);\ L \le K;\ 0{<}L]\!] \Longrightarrow K \otimes L = K$
⟨*proof*⟩

lemma *InfCard-cmult-eq*: $[\![InfCard(K);\ InfCard(L)]\!] \Longrightarrow K \otimes L = K \cup L$
⟨*proof*⟩

lemma *InfCard-cdouble-eq*: $InfCard(K) \Longrightarrow K \oplus K = K$
⟨*proof*⟩

lemma *InfCard-le-cadd-eq*: $[\![InfCard(K);\ \ L \leq K]\!] \Longrightarrow K \oplus L = K$
⟨*proof*⟩

lemma *InfCard-cadd-eq*: $[\![InfCard(K);\ \ InfCard(L)]\!] \Longrightarrow K \oplus L = K \cup L$
⟨*proof*⟩

34.6 For Every Cardinal Number There Exists A Greater One

This result is Kunen's Theorem 10.16, which would be trivial using AC

lemma *Ord-jump-cardinal*: $Ord(jump\text{-}cardinal(K))$
⟨*proof*⟩

lemma *jump-cardinal-iff*:
$i \in jump\text{-}cardinal(K) \longleftrightarrow$
$(\exists\, r\ X.\ r \subseteq K{*}K \wedge X \subseteq K \wedge well\text{-}ord(X,r) \wedge i = ordertype(X,r))$
⟨*proof*⟩

lemma *K-lt-jump-cardinal*: $Ord(K) \Longrightarrow K < jump\text{-}cardinal(K)$
⟨*proof*⟩

lemma *Card-jump-cardinal-lemma*:
$[\![well\text{-}ord(X,r);\ \ r \subseteq K * K;\ \ X \subseteq K;$
$f \in bij(ordertype(X,r), jump\text{-}cardinal(K))]\!]$
$\Longrightarrow jump\text{-}cardinal(K) \in jump\text{-}cardinal(K)$
⟨*proof*⟩

lemma *Card-jump-cardinal*: $Card(jump\text{-}cardinal(K))$
⟨*proof*⟩

34.7 Basic Properties of Successor Cardinals

lemma *csucc-basic*: $Ord(K) \Longrightarrow Card(csucc(K)) \wedge K < csucc(K)$
⟨*proof*⟩

lemmas *Card-csucc* = *csucc-basic* [*THEN conjunct1*]

lemmas *lt-csucc* = *csucc-basic* [*THEN conjunct2*]

lemma *Ord-0-lt-csucc*: $Ord(K) \Longrightarrow 0 < csucc(K)$
⟨*proof*⟩

lemma *csucc-le*: $[\![Card(L);\ \ K{<}L]\!] \Longrightarrow csucc(K) \leq L$

⟨*proof*⟩

lemma *lt-csucc-iff*: ⟦*Ord*(*i*); *Card*(*K*)⟧ ⟹ *i* < *csucc*(*K*) ⟷ |*i*| ≤ *K*
⟨*proof*⟩

lemma *Card-lt-csucc-iff*:
⟦*Card*(*K'*); *Card*(*K*)⟧ ⟹ *K'* < *csucc*(*K*) ⟷ *K'* ≤ *K*
⟨*proof*⟩

lemma *InfCard-csucc*: *InfCard*(*K*) ⟹ *InfCard*(*csucc*(*K*))
⟨*proof*⟩

34.7.1 Removing elements from a finite set decreases its cardinality

lemma *Finite-imp-cardinal-cons* [*simp*]:
assumes *FA*: *Finite*(*A*) **and** *a*: *a*∉*A* **shows** |*cons*(*a*,*A*)| = *succ*(|*A*|)
⟨*proof*⟩

lemma *Finite-imp-succ-cardinal-Diff*:
⟦*Finite*(*A*); *a* ∈ *A*⟧ ⟹ *succ*(|*A*−{*a*}|) = |*A*|
⟨*proof*⟩

lemma *Finite-imp-cardinal-Diff*: ⟦*Finite*(*A*); *a* ∈ *A*⟧ ⟹ |*A*−{*a*}| < |*A*|
⟨*proof*⟩

lemma *Finite-cardinal-in-nat* [*simp*]: *Finite*(*A*) ⟹ |*A*| ∈ *nat*
⟨*proof*⟩

lemma *card-Un-Int*:
⟦*Finite*(*A*); *Finite*(*B*)⟧ ⟹ |*A*| #+ |*B*| = |*A* ∪ *B*| #+ |*A* ∩ *B*|
⟨*proof*⟩

lemma *card-Un-disjoint*:
⟦*Finite*(*A*); *Finite*(*B*); *A* ∩ *B* = *0*⟧ ⟹ |*A* ∪ *B*| = |*A*| #+ |*B*|
⟨*proof*⟩

lemma *card-partition*:
assumes *FC*: *Finite*(*C*)
shows
Finite (⋃ *C*) ⟹
(∀ *c*∈*C*. |*c*| = *k*) ⟹
(∀ *c1* ∈ *C*. ∀ *c2* ∈ *C*. *c1* ≠ *c2* ⟶ *c1* ∩ *c2* = *0*) ⟹
k #∗ |*C*| = |⋃ *C*|
⟨*proof*⟩

34.7.2 Theorems by Krzysztof Grabczewski, proofs by lcp

lemmas *nat-implies-well-ord* = *nat-into-Ord* [*THEN well-ord-Memrel*]

lemma *nat-sum-eqpoll-sum*:
  assumes *m*: $m \in nat$ **and** *n*: $n \in nat$ **shows** $m + n \approx m \#+ n$
⟨*proof*⟩

lemma *Ord-subset-natD* [*rule-format*]: $Ord(i) \Longrightarrow i \subseteq nat \Longrightarrow i \in nat \mid i{=}nat$
⟨*proof*⟩

lemma *Ord-nat-subset-into-Card*: $[\![Ord(i);\ i \subseteq nat]\!] \Longrightarrow Card(i)$
  ⟨*proof*⟩

end

35 Main ZF Theory: Everything Except AC

theory *ZF* **imports** *List IntDiv CardinalArith* **begin**

35.1 Iteration of the function F

consts *iterates* :: $[i \Rightarrow i, i, i] \Rightarrow i$ (‹(-^- '(-'))› [60,1000,1000] 60)

primrec
    $F^0\ (x) = x$
    $F^(succ(n))\ (x) = F(F^n\ (x))$

definition
  iterates-omega :: $[i \Rightarrow i, i] \Rightarrow i$ (‹(-^ω '(-'))› [60,1000] 60) **where**
    $F^\omega\ (x) \equiv \bigcup n{\in}nat.\ F^n\ (x)$

lemma *iterates-triv*:
    $[\![n{\in}nat;\ \ F(x) = x]\!] \Longrightarrow F^n\ (x) = x$
⟨*proof*⟩

lemma *iterates-type* [*TC*]:
    $[\![n \in nat;\ \ a \in A;\ \bigwedge x.\ x \in A \Longrightarrow F(x) \in A]\!]$
     $\Longrightarrow F^n\ (a) \in A$
⟨*proof*⟩

lemma *iterates-omega-triv*:
    $F(x) = x \Longrightarrow F^\omega\ (x) = x$
⟨*proof*⟩

lemma *Ord-iterates* [*simp*]:
    $[\![n{\in}nat;\ \ \bigwedge i.\ Ord(i) \Longrightarrow Ord(F(i));\ \ Ord(x)]\!]$
     $\Longrightarrow Ord(F^n\ (x))$
⟨*proof*⟩

lemma *iterates-commute*: $n \in nat \Longrightarrow F(F^n\ (x)) = F^n\ (F(x))$
⟨*proof*⟩

35.2 Transfinite Recursion

Transfinite recursion for definitions based on the three cases of ordinals

definition
transrec3 :: [*i*, *i*, [*i*,*i*]⇒*i*, [*i*,*i*]⇒*i*] ⇒*i* **where**
transrec3(*k*, *a*, *b*, *c*) ≡
transrec(*k*, λ*x* *r*.
if *x*=*0* *then* *a*
else if *Limit*(*x*) *then* *c*(*x*, λ*y*∈*x*. *r*‘*y*)
else *b*(*Arith.pred*(*x*), *r* ‘ *Arith.pred*(*x*)))

lemma *transrec3-0* [*simp*]: *transrec3*(*0*,*a*,*b*,*c*) = *a*
⟨*proof*⟩

lemma *transrec3-succ* [*simp*]:
transrec3(*succ*(*i*),*a*,*b*,*c*) = *b*(*i*, *transrec3*(*i*,*a*,*b*,*c*))
⟨*proof*⟩

lemma *transrec3-Limit*:
Limit(*i*) ⟹
transrec3(*i*,*a*,*b*,*c*) = *c*(*i*, λ*j*∈*i*. *transrec3*(*j*,*a*,*b*,*c*))
⟨*proof*⟩

⟨*ML*⟩

end

36 The Axiom of Choice

theory *AC* **imports** *ZF* **begin**

This definition comes from Halmos (1960), page 59.

axiomatization where
AC: ⟦*a* ∈ *A*; ⋀*x*. *x* ∈ *A* ⟹ (∃ *y*. *y* ∈ *B*(*x*))⟧ ⟹ ∃ *z*. *z* ∈ *Pi*(*A*,*B*)

lemma *AC-Pi*: ⟦⋀*x*. *x* ∈ *A* ⟹ (∃ *y*. *y* ∈ *B*(*x*))⟧ ⟹ ∃ *z*. *z* ∈ *Pi*(*A*,*B*)
⟨*proof*⟩

lemma *AC-ball-Pi*: ∀ *x* ∈ *A*. ∃ *y*. *y* ∈ *B*(*x*) ⟹ ∃ *y*. *y* ∈ *Pi*(*A*,*B*)
⟨*proof*⟩

lemma *AC-Pi-Pow*: ∃ *f*. *f* ∈ (∏ *X* ∈ *Pow*(*C*)−{*0*}. *X*)
⟨*proof*⟩

lemma *AC-func*:

$[\![\bigwedge x.\ x \in A \Longrightarrow (\exists\, y.\ y \in x)]\!] \Longrightarrow \exists f \in A{->}\bigcup(A).\ \forall x \in A.\ f`x \in x$
⟨*proof*⟩

lemma *non-empty-family*: $[\![0 \notin A;\ \ x \in A]\!] \Longrightarrow \exists\, y.\ y \in x$
⟨*proof*⟩

lemma *AC-func0*: $0 \notin A \Longrightarrow \exists f \in A{->}\bigcup(A).\ \forall x \in A.\ f`x \in x$
⟨*proof*⟩

lemma *AC-func-Pow*: $\exists f \in (Pow(C){-}\{0\}) \mathrel{->} C.\ \forall x \in Pow(C){-}\{0\}.\ f`x \in x$
⟨*proof*⟩

lemma *AC-Pi0*: $0 \notin A \Longrightarrow \exists f.\ f \in (\prod x \in A.\ x)$
⟨*proof*⟩

end

37 Zorn's Lemma

theory *Zorn* **imports** *OrderArith AC Inductive* **begin**

Based upon the unpublished article "Towards the Mechanization of the Proofs of Some Classical Theorems of Set Theory," by Abrial and Laffitte.

definition
Subset-rel :: $i \Rightarrow i$ **where**
$Subset\text{-}rel(A) \equiv \{z \in A*A\ .\ \exists x\ y.\ z=\langle x,y\rangle \wedge x<=y \wedge x \neq y\}$

definition
chain :: $i \Rightarrow i$ **where**
$chain(A) \equiv \{F \in Pow(A).\ \forall X{\in}F.\ \forall Y{\in}F.\ X<=Y \mid Y<=X\}$

definition
super :: $[i,i] \Rightarrow i$ **where**
$super(A,c) \equiv \{d \in chain(A).\ c<=d \wedge c \neq d\}$

definition
maxchain :: $i \Rightarrow i$ **where**
$maxchain(A) \equiv \{c \in chain(A).\ super(A,c)=0\}$

definition
increasing :: $i \Rightarrow i$ **where**
$increasing(A) \equiv \{f \in Pow(A){->}Pow(A).\ \forall x.\ x<=A \longrightarrow x<=f`x\}$

Lemma for the inductive definition below

lemma *Union-in-Pow*: $Y \in Pow(Pow(A)) \Longrightarrow \bigcup(Y) \in Pow(A)$
⟨*proof*⟩

We could make the inductive definition conditional on $next \in increasing(S)$ but instead we make this a side-condition of an introduction rule. Thus

the induction rule lets us assume that condition! Many inductive proofs are therefore unconditional.

consts
TFin :: $[i,i]\Rightarrow i$

inductive
domains *TFin*(*S*,*next*) $\subseteq$ *Pow*(*S*)
intros
nextI: $[\![x \in TFin(S,next);\ next \in increasing(S)]\!] \Longrightarrow next`x \in TFin(S,next)$

Pow-UnionI: $Y \in Pow(TFin(S,next)) \Longrightarrow \bigcup(Y) \in TFin(S,next)$

monos *Pow-mono*
con-defs *increasing-def*
type-intros *CollectD1* [*THEN apply-funtype*] *Union-in-Pow*

37.1 Mathematical Preamble

lemma *Union-lemma0*: $(\forall x{\in}C.\ x{<=}A \mid B{<=}x) \Longrightarrow \bigcup(C){<=}A \mid B{<=}\bigcup(C)$
⟨*proof*⟩

lemma *Inter-lemma0*:
$[\![c \in C; \forall x{\in}C.\ A{<=}x \mid x{<=}B]\!] \Longrightarrow A \subseteq \bigcap(C) \mid \bigcap(C) \subseteq B$
⟨*proof*⟩

37.2 The Transfinite Construction

lemma *increasingD1*: $f \in increasing(A) \Longrightarrow f \in Pow(A){-}{>}Pow(A)$
⟨*proof*⟩

lemma *increasingD2*: $[\![f \in increasing(A); x{<=}A]\!] \Longrightarrow x \subseteq f`x$
⟨*proof*⟩

lemmas *TFin-UnionI* = *PowI* [*THEN TFin.Pow-UnionI*]

lemmas *TFin-is-subset* = *TFin.dom-subset* [*THEN subsetD*, *THEN PowD*]

Structural induction on *TFin*(*S*, *next*)

lemma *TFin-induct*:
$[\![n \in TFin(S,next);$
$\bigwedge x.\ [\![x \in TFin(S,next);\ P(x);\ next \in increasing(S)]\!] \Longrightarrow P(next`x);$
$\bigwedge Y.\ [\![Y \subseteq TFin(S,next);\ \forall y{\in}Y.\ P(y)]\!] \Longrightarrow P(\bigcup(Y))$
$]\!] \Longrightarrow P(n)$
⟨*proof*⟩

37.3 Some Properties of the Transfinite Construction

lemmas *increasing-trans* = *subset-trans* [*OF - increasingD2*,

OF - - TFin-is-subset]

Lemma 1 of section 3.1

lemma *TFin-linear-lemma1*:
⟦$n \in TFin(S,next)$; $m \in TFin(S,next)$;
$\forall x \in TFin(S,next)$. $x<=m \longrightarrow x=m \mid next`x<=m$⟧
$\Longrightarrow n<=m \mid next`m<=n$
⟨*proof*⟩

Lemma 2 of section 3.2. Interesting in its own right! Requires $next \in increasing(S)$ in the second induction step.

lemma *TFin-linear-lemma2*:
⟦$m \in TFin(S,next)$; $next \in increasing(S)$⟧
$\Longrightarrow \forall n \in TFin(S,next).\ n<=m \longrightarrow n=m \mid next`n \subseteq m$
⟨*proof*⟩

a more convenient form for Lemma 2

lemma *TFin-subsetD*:
⟦$n<=m$; $m \in TFin(S,next)$; $n \in TFin(S,next)$; $next \in increasing(S)$⟧
$\Longrightarrow n=m \mid next`n \subseteq m$
⟨*proof*⟩

Consequences from section 3.3 – Property 3.2, the ordering is total

lemma *TFin-subset-linear*:
⟦$m \in TFin(S,next)$; $n \in TFin(S,next)$; $next \in increasing(S)$⟧
$\Longrightarrow n \subseteq m \mid m<=n$
⟨*proof*⟩

Lemma 3 of section 3.3

lemma *equal-next-upper*:
⟦$n \in TFin(S,next)$; $m \in TFin(S,next)$; $m = next`m$⟧ $\Longrightarrow n \subseteq m$
⟨*proof*⟩

Property 3.3 of section 3.3

lemma *equal-next-Union*:
⟦$m \in TFin(S,next)$; $next \in increasing(S)$⟧
$\Longrightarrow m = next`m <-> m = \bigcup(TFin(S,next))$
⟨*proof*⟩

37.4 Hausdorff's Theorem: Every Set Contains a Maximal Chain

NOTE: We assume the partial ordering is $\subseteq$, the subset relation!

* Defining the "next" operation for Hausdorff's Theorem *

lemma *chain-subset-Pow*: $chain(A) \subseteq Pow(A)$
⟨*proof*⟩

lemma *super-subset-chain*: $super(A,c) \subseteq chain(A)$
⟨*proof*⟩

lemma *maxchain-subset-chain*: $maxchain(A) \subseteq chain(A)$
⟨*proof*⟩

lemma *choice-super*:
$[\![ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X);\ X \in chain(S);\ \ X \notin maxchain(S)]\!]$
$\Longrightarrow ch\ `\ super(S,X) \in super(S,X)$
⟨*proof*⟩

lemma *choice-not-equals*:
$[\![ch \in (\prod X \in Pow(chain(S)) - \{0\}.\ X);\ X \in chain(S);\ \ X \notin maxchain(S)]\!]$
$\Longrightarrow ch\ `\ super(S,X) \neq X$
⟨*proof*⟩

This justifies Definition 4.4

lemma *Hausdorff-next-exists*:
$ch \in (\prod X \in Pow(chain(S))-\{0\}.\ X) \Longrightarrow$
$\exists next \in increasing(S).\ \forall X \in Pow(S).$
$next`X = if(X \in chain(S)-maxchain(S),\ ch`super(S,X),\ X)$
⟨*proof*⟩

Lemma 4

lemma *TFin-chain-lemma4*:
$[\![c \in TFin(S,next);$
$ch \in (\prod X \in Pow(chain(S))-\{0\}.\ X);$
$next \in increasing(S);$
$\forall X \in Pow(S).\ next`X =$
$if(X \in chain(S)-maxchain(S),\ ch`super(S,X),\ X)]\!]$
$\Longrightarrow c \in chain(S)$
⟨*proof*⟩

theorem *Hausdorff*: $\exists c.\ c \in maxchain(S)$
⟨*proof*⟩

37.5 Zorn's Lemma: If All Chains in S Have Upper Bounds In S, then S contains a Maximal Element

Used in the proof of Zorn's Lemma

lemma *chain-extend*:
$[\![c \in chain(A);\ \ z \in A;\ \ \forall x \in c.\ x<=z]\!] \Longrightarrow cons(z,c) \in chain(A)$
⟨*proof*⟩

lemma *Zorn*: $\forall c \in chain(S).\ \bigcup(c) \in S \Longrightarrow \exists y \in S.\ \forall z \in S.\ y<=z \longrightarrow y=z$
⟨*proof*⟩

Alternative version of Zorn's Lemma

theorem *Zorn2*:
$\forall c \in chain(S).\ \exists y \in S.\ \forall x \in c.\ x \subseteq y \Longrightarrow \exists y \in S.\ \forall z \in S.\ y{<}{=}z \longrightarrow y{=}z$
⟨*proof*⟩

37.6 Zermelo's Theorem: Every Set can be Well-Ordered

Lemma 5

lemma *TFin-well-lemma5*:
$[\![n \in TFin(S,next);\ \ Z \subseteq TFin(S,next);\ \ z{:}Z;\ \ \neg \bigcap(Z) \in Z]\!]$
$\Longrightarrow \forall m \in Z.\ n \subseteq m$
⟨*proof*⟩

Well-ordering of $TFin(S,\ next)$

lemma *well-ord-TFin-lemma*: $[\![Z \subseteq TFin(S,next);\ \ z \in Z]\!] \Longrightarrow \bigcap(Z) \in Z$
⟨*proof*⟩

This theorem just packages the previous result

lemma *well-ord-TFin*:
$next \in increasing(S)$
$\Longrightarrow well\text{-}ord(TFin(S,next),\ Subset\text{-}rel(TFin(S,next)))$
⟨*proof*⟩

* Defining the "next" operation for Zermelo's Theorem *

lemma *choice-Diff*:
$[\![ch \in (\prod X \in Pow(S) - \{0\}.\ X);\ \ X \subseteq S;\ \ X{\neq}S]\!] \Longrightarrow ch\ `\ (S{-}X) \in S{-}X$
⟨*proof*⟩

This justifies Definition 6.1

lemma *Zermelo-next-exists*:
$ch \in (\prod X \in Pow(S){-}\{0\}.\ X) \Longrightarrow$
$\exists next \in increasing(S).\ \forall X \in Pow(S).$
$next`X = (if\ X{=}S\ then\ S\ else\ cons(ch`(S{-}X),\ X))$
⟨*proof*⟩

The construction of the injection

lemma *choice-imp-injection*:
$[\![ch \in (\prod X \in Pow(S){-}\{0\}.\ X);$
$next \in increasing(S);$
$\forall X \in Pow(S).\ next`X = if(X{=}S,\ S,\ cons(ch`(S{-}X),\ X))]\!]$
$\Longrightarrow (\lambda\ x \in S.\ \bigcup(\{y \in TFin(S,next).\ x \notin y\}))$
$\in inj(S,\ TFin(S,next) - \{S\})$
⟨*proof*⟩

The wellordering theorem

theorem *AC-well-ord*: $\exists r.\ well\text{-}ord(S,r)$
⟨*proof*⟩

37.7 Zorn's Lemma for Partial Orders

Reimported from HOL by Clemens Ballarin.

definition *Chain* :: $i \Rightarrow i$ **where**
$Chain(r) = \{A \in Pow(field(r)).\ \forall a{\in}A.\ \forall b{\in}A.\ \langle a, b\rangle \in r \mid \langle b, a\rangle \in r\}$

lemma *mono-Chain*:
$r \subseteq s \Longrightarrow Chain(r) \subseteq Chain(s)$
⟨*proof*⟩

theorem *Zorn-po*:
assumes *po*: $Partial\text{-}order(r)$
and *u*: $\forall C{\in}Chain(r).\ \exists u{\in}field(r).\ \forall a{\in}C.\ \langle a, u\rangle \in r$
shows $\exists m{\in}field(r).\ \forall a{\in}field(r).\ \langle m, a\rangle \in r \longrightarrow a = m$
⟨*proof*⟩

end

38 Cardinal Arithmetic Using AC

theory *Cardinal-AC* **imports** *CardinalArith Zorn* **begin**

38.1 Strengthened Forms of Existing Theorems on Cardinals

lemma *cardinal-eqpoll*: $|A| \approx A$
⟨*proof*⟩

The theorem $||A|| = |A|$

lemmas *cardinal-idem* = *cardinal-eqpoll* [*THEN cardinal-cong*, *simp*]

lemma *cardinal-eqE*: $|X| = |Y| \Longrightarrow X \approx Y$
⟨*proof*⟩

lemma *cardinal-eqpoll-iff*: $|X| = |Y| \longleftrightarrow X \approx Y$
⟨*proof*⟩

lemma *cardinal-disjoint-Un*:
$[\![|A|{=}|B|;\ |C|{=}|D|;\ A \cap C = 0;\ B \cap D = 0]\!]$
$\Longrightarrow |A \cup C| = |B \cup D|$
⟨*proof*⟩

lemma *lepoll-imp-cardinal-le*: $A \lesssim B \Longrightarrow |A| \le |B|$
⟨*proof*⟩

lemma *cadd-assoc*: $(i \oplus j) \oplus k = i \oplus (j \oplus k)$
⟨*proof*⟩

lemma *cmult-assoc*: $(i \otimes j) \otimes k = i \otimes (j \otimes k)$

⟨*proof*⟩

lemma *cadd-cmult-distrib*: $(i \oplus j) \otimes k = (i \otimes k) \oplus (j \otimes k)$
⟨*proof*⟩

lemma *InfCard-square-eq*: $InfCard(|A|) \Longrightarrow A*A \approx A$
⟨*proof*⟩

38.2 The relationship between cardinality and le-pollence

lemma *Card-le-imp-lepoll*:
assumes $|A| \le |B|$ **shows** $A \lesssim B$
⟨*proof*⟩

lemma *le-Card-iff*: $Card(K) \Longrightarrow |A| \le K \longleftrightarrow A \lesssim K$
⟨*proof*⟩

lemma *cardinal-0-iff-0* [*simp*]: $|A| = 0 \longleftrightarrow A = 0$
⟨*proof*⟩

lemma *cardinal-lt-iff-lesspoll*:
assumes i: $Ord(i)$ **shows** $i < |A| \longleftrightarrow i \prec A$
⟨*proof*⟩

lemma *cardinal-le-imp-lepoll*: $i \le |A| \Longrightarrow i \lesssim A$
⟨*proof*⟩

38.3 Other Applications of AC

lemma *surj-implies-inj*:
assumes f: $f \in surj(X,Y)$ **shows** $\exists g.\ g \in inj(Y,X)$
⟨*proof*⟩

Kunen's Lemma 10.20

lemma *surj-implies-cardinal-le*:
assumes f: $f \in surj(X,Y)$ **shows** $|Y| \le |X|$
⟨*proof*⟩

Kunen's Lemma 10.21

lemma *cardinal-UN-le*:
assumes K: $InfCard(K)$
shows $(\bigwedge i.\ i{\in}K \Longrightarrow |X(i)| \le K) \Longrightarrow |\bigcup i{\in}K.\ X(i)| \le K$
⟨*proof*⟩

The same again, using *csucc*

lemma *cardinal-UN-lt-csucc*:
$[\![InfCard(K);\ \bigwedge i.\ i{\in}K \Longrightarrow |X(i)| < csucc(K)]\!]$
$\Longrightarrow |\bigcup i{\in}K.\ X(i)| < csucc(K)$
⟨*proof*⟩

The same again, for a union of ordinals. In use, j(i) is a bit like rank(i), the least ordinal j such that i:Vfrom(A,j).

lemma *cardinal-UN-Ord-lt-csucc*:
$[\![InfCard(K);\ \bigwedge i.\ i{\in}K \Longrightarrow j(i) < csucc(K)]\!]$
$\Longrightarrow (\bigcup i{\in}K.\ j(i)) < csucc(K)$
⟨*proof*⟩

38.4 The Main Result for Infinite-Branching Datatypes

As above, but the index set need not be a cardinal. Work backwards along the injection from W into K, given that $W \neq 0$.

lemma *inj-UN-subset*:
assumes *f*: $f \in inj(A,B)$ **and** *a*: $a \in A$
shows $(\bigcup x{\in}A.\ C(x)) \subseteq (\bigcup y{\in}B.\ C(\mathit{if}\ y \in range(f)\ \mathit{then}\ converse(f)`y\ \mathit{else}\ a))$
⟨*proof*⟩

theorem *le-UN-Ord-lt-csucc*:
assumes *IK*: $InfCard(K)$ **and** *WK*: $|W| \leq K$ **and** *j*: $\bigwedge w.\ w{\in}W \Longrightarrow j(w) < csucc(K)$
shows $(\bigcup w{\in}W.\ j(w)) < csucc(K)$
⟨*proof*⟩

end

39 Infinite-Branching Datatype Definitions

theory *InfDatatype* **imports** *Datatype Univ Finite Cardinal-AC* **begin**

lemmas *fun-Limit-VfromE* =
Limit-VfromE [*OF apply-funtype InfCard-csucc* [*THEN InfCard-is-Limit*]]

lemma *fun-Vcsucc-lemma*:
assumes *f*: $f \in D \to Vfrom(A,csucc(K))$ **and** *DK*: $|D| \leq K$ **and** *ICK*: $InfCard(K)$
shows $\exists j.\ f \in D \to Vfrom(A,j) \wedge j < csucc(K)$
⟨*proof*⟩

lemma *subset-Vcsucc*:
$[\![D \subseteq Vfrom(A,csucc(K));\ |D| \leq K;\ InfCard(K)]\!]$
$\Longrightarrow \exists j.\ D \subseteq Vfrom(A,j) \wedge j < csucc(K)$
⟨*proof*⟩

lemma *fun-Vcsucc*:
$[\![|D| \leq K;\ InfCard(K);\ D \subseteq Vfrom(A,csucc(K))]\!] \Longrightarrow$
$D \to Vfrom(A,csucc(K)) \subseteq Vfrom(A,csucc(K))$
⟨*proof*⟩

lemma *fun-in-Vcsucc*:
$[\![f\colon D \mathrel{->} Vfrom(A,\ csucc(K));\ \ |D| \le K;\ \ InfCard(K);$
$D \subseteq Vfrom(A,csucc(K))]\!]$
$\Longrightarrow f\colon Vfrom(A,csucc(K))$
⟨*proof*⟩

Remove $\subseteq$ from the rule above

lemmas *fun-in-Vcsucc'* = *fun-in-Vcsucc* [*OF - - - subsetI*]

lemma *Card-fun-Vcsucc*:
$InfCard(K) \Longrightarrow K \mathrel{->} Vfrom(A,csucc(K)) \subseteq Vfrom(A,csucc(K))$
⟨*proof*⟩

lemma *Card-fun-in-Vcsucc*:
$[\![f\colon K \mathrel{->} Vfrom(A,\ csucc(K));\ \ InfCard(K)]\!] \Longrightarrow f\colon Vfrom(A,csucc(K))$
⟨*proof*⟩

lemma *Limit-csucc*: $InfCard(K) \Longrightarrow Limit(csucc(K))$
⟨*proof*⟩

lemmas *Pair-in-Vcsucc* = *Pair-in-VLimit* [*OF - - Limit-csucc*]
lemmas *Inl-in-Vcsucc* = *Inl-in-VLimit* [*OF - Limit-csucc*]
lemmas *Inr-in-Vcsucc* = *Inr-in-VLimit* [*OF - Limit-csucc*]
lemmas *zero-in-Vcsucc* = *Limit-csucc* [*THEN zero-in-VLimit*]
lemmas *nat-into-Vcsucc* = *nat-into-VLimit* [*OF - Limit-csucc*]

lemmas *InfCard-nat-Un-cardinal* = *InfCard-Un* [*OF InfCard-nat Card-cardinal*]

lemmas *le-nat-Un-cardinal* =
Un-upper2-le [*OF Ord-nat Card-cardinal* [*THEN Card-is-Ord*]]

lemmas *UN-upper-cardinal* = *UN-upper* [*THEN subset-imp-lepoll*, *THEN lepoll-imp-cardinal-le*]

lemmas *Data-Arg-intros* =
SigmaI InlI InrI
Pair-in-univ Inl-in-univ Inr-in-univ
zero-in-univ A-into-univ nat-into-univ UnCI

lemmas *inf-datatype-intros* =
InfCard-nat InfCard-nat-Un-cardinal
Pair-in-Vcsucc Inl-in-Vcsucc Inr-in-Vcsucc

zero-in-Vcsucc A-into-Vfrom nat-into-Vcsucc
Card-fun-in-Vcsucc fun-in-Vcsucc′ UN-I

end
theory *ZFC* **imports** *ZF InfDatatype*
begin

end